Automates cellulaires probabilistes et mesures spécifiques sur des espaces symboliques

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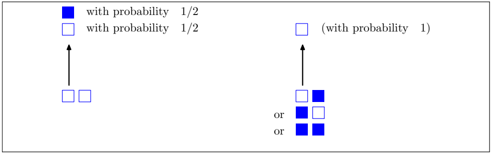
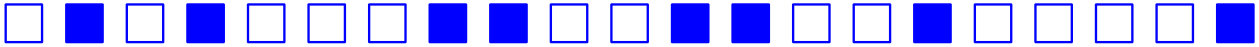
Thèse effectuée sous la direction de Jean Mairesse

Vendredi 22 novembre 2013

Example of PCA





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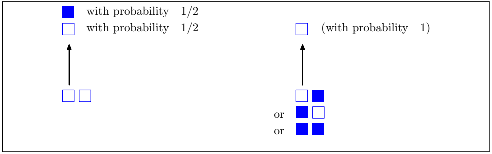

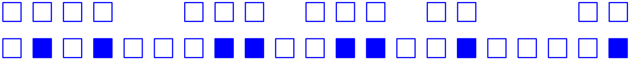

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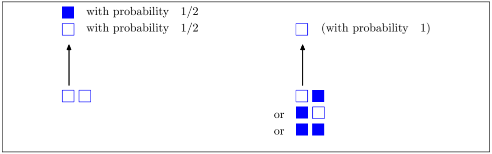

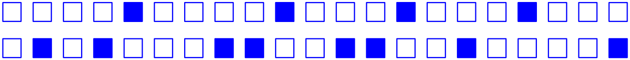

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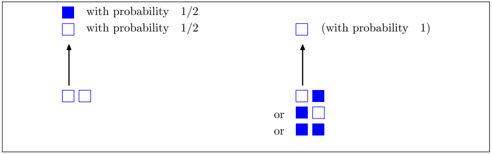

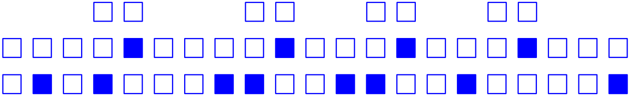

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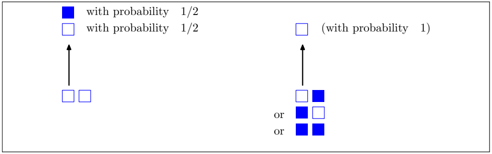

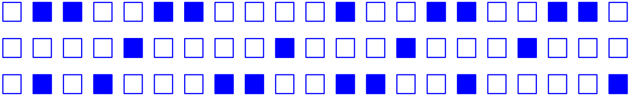

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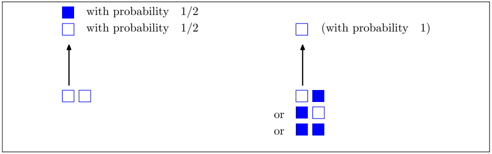
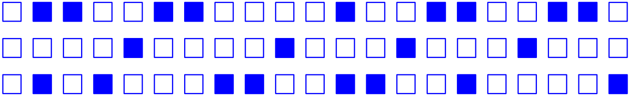


PCA of set of cells $E = \mathbb{Z}$,



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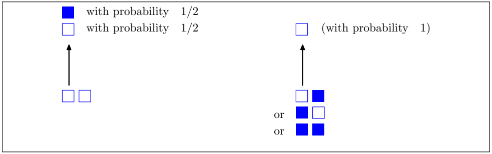
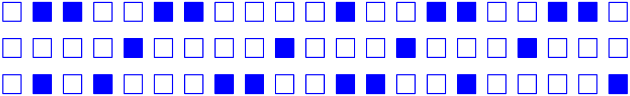


PCA of set of cells $E = \mathbb{Z}$, alphabet $\mathcal{A} = \{\blacksquare, \square\}$,



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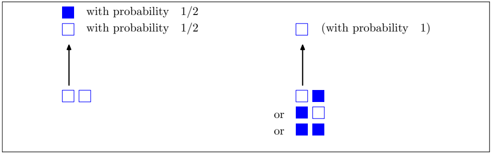
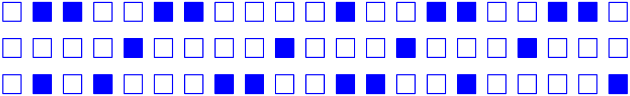


PCA of set of cells $E = \mathbb{Z}$, alphabet $\mathcal{A} = \{■, □\}$, neighbourhood $\mathcal{N} = \{0, 1\}$.



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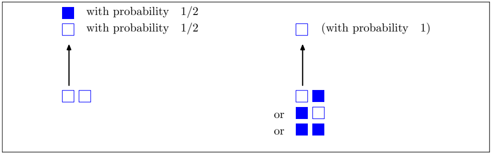

PCA of set of cells $E = \mathbb{Z}$, alphabet $\mathcal{A} = \{\blacksquare, \square\}$, neighbourhood $\mathcal{N} = \{0, 1\}$.

Local function:

$$f : \{\blacksquare, \square\}^2 \to \mathcal{M}(\{\blacksquare, \square\})$$

defined by:

$$f(\square\square) = \frac{1}{2}\delta_{\square} + \frac{1}{2}\delta_{\blacksquare}$$

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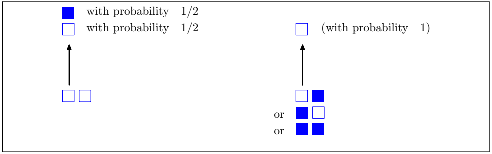

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Local function:

$$f : \{\blacksquare, \square\}^2 \to \mathcal{M}(\{\blacksquare, \square\})$$

Global function:

$$F : \mathcal{M}(\{\blacksquare, \square\}^{\mathbb{Z}}) \to \mathcal{M}(\{\blacksquare, \square\}^{\mathbb{Z}})$$

$$\mu \mapsto \mu F$$

Definition of PCA

Let $\mathcal{A}$ be a finite set called the alphabet.

A PCA F of set of cells $E = \mathbb{Z}^d$ is defined by

- a finite neighbourhood $\mathcal{N} \subset E$,
- a local function $f : \mathcal{A}^{\mathcal{N}} \to \mathcal{M}(\mathcal{A})$.

From the configuration $(x_k)_{k \in E} \in \mathcal{A}^E$, cell k is updated by the symbol y with probability:

$$f((x_{k+v})_{v \in \mathcal{N}})(y),$$

simultaneously and independently of the other cells.

Motivations for the study of PCA.

$$\text{PCA} = \begin{cases} \text{synchronous analogous of interacting particle systems,} \\ \text{natural extension of deterministic CA.} \end{cases}$$

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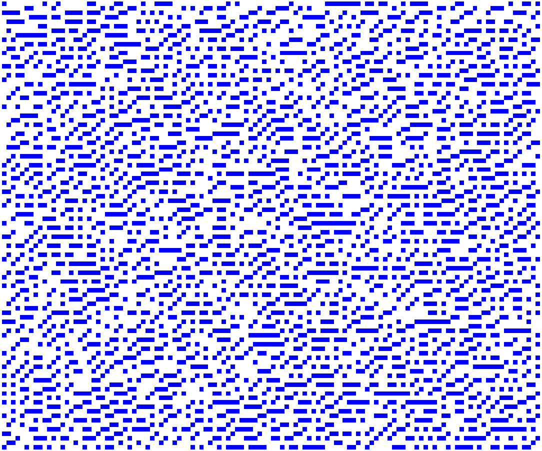
- Modelling tool in physics and in the life sciences.
- Link with different problems in probability, combinatorics, symbolic dynamics.

Back to the example



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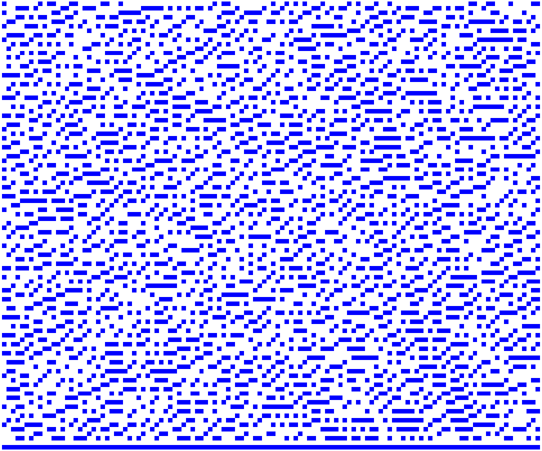

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The system **forgets** its initial configuration. We say it is ergodic.

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Ergodicity

A PCA F on $\mathcal{A}^E$ is ergodic if:

- it has a **unique invariant measure** $\pi \in \mathcal{M}(\mathcal{A}^E)$, such that $\pi F = \pi$,
- for any initial measure $\mu \in \mathcal{M}(\mathcal{A}^E)$, the sequence of iterates $(\mu F^n)_{n \geq 0}$ **converges** weakly to π.

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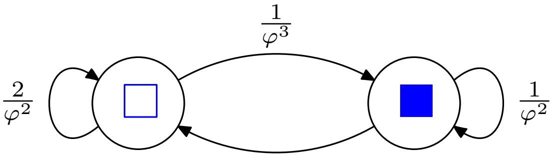

with $\quad \pi_{\square} = \frac{\varphi^2}{1+\varphi^2} \quad$ and $\quad \pi_{\blacksquare} = \frac{1}{1+\varphi^2} \qquad \left(\varphi = \frac{1+\sqrt{5}}{2}\right)$.

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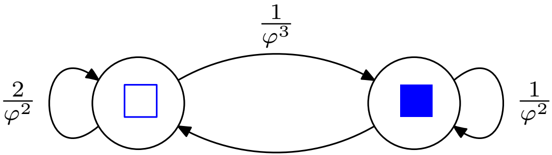

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But for general PCA, the ergodicity is difficult to determine, and we have no expression of the invariant measure(s)!

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3. Measures of maximal entropy of subshift of finite type (SFT).
   They are also invariant measures of a well-suited PCA.

Plan

1. Ergodicity and perfect sampling
   - Undecidability of the ergodicity
   - Perfect sampling
2. PCA having a specific behaviour
   - Bernoulli invariant measures
   - Density classification
3. Measures on subshift of finite type
   - One-dimensional SFT and the Parry measure
   - Link with PCA

Ergodicity of deterministic CA

A deterministic CA $F : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ is nilpotent if there exists $\alpha \in \mathcal{A}$ such that: $\forall x \in \mathcal{A}^{\mathbb{Z}^d}, \exists n \in \mathbb{N}, F^n(x) = \alpha^{\mathbb{Z}^d}$.

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Proposition [Bušić-Mairesse-M., STACS 2011 & Adv. Appl. Proba]

For deterministic CA, ergodicity $\Leftrightarrow$ nilpotency.

Proof. $\Leftarrow$ is easy ; $\Rightarrow$ in two steps:

1. the unique invariant measure has to be a measure concentrated on a monochromatic configuration $\alpha^{\mathbb{Z}^d}$,
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Corollary (with [Kari 1992])

The ergodicity of one-dimensional deterministic CA (and hence of PCA) is undecidable.

Perfect sampling for PCA

Let F be an ergodic PCA of invariant measure π. In general, we have no explicit description of π.

Perfect sampling of π: probabilistic algorithm returning a sequence $a_1 \dots a_n$ with *exactly* the probability it has to appear under the measure π.

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Idea: adapt the coupling from the past algorithm [Propp-Wilson 1996], with the introduction of a bounding process called the envelope PCA.

Update function of a PCA

A way to run a PCA (on $\mathcal{A} = \{0,1\}$) from configuration $x \in \mathcal{A}^{\mathbb{Z}}$:

- generate for each cell k independently and uniformly a random number r_k in $[0,1]$,
- choose the new state of the cell k to be

$$\mathbf{0} \text{ if } r_k < f((x_{k+v})_{v\in\mathcal{N}})(0), \text{ and } \mathbf{1} \text{ otherwise.}$$

$$\ldots \quad x_{-3} \quad x_{-2} \quad x_{-1} \quad x_0 \quad x_1 \quad x_2 \quad x_3 \quad \ldots$$

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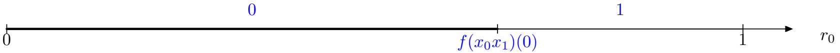

$$\mathbf{y_0}$$

$$\begin{array}{ccccccccc} \dots & x_{-3} & x_{-2} & x_{-1} & x_0 & x_1 & x_2 & x_3 & \dots \\ \dots & r_{-3} & r_{-2} & r_{-1} & r_0 & r_1 & r_2 & r_3 & \dots \end{array}$$

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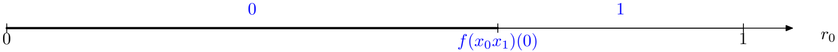


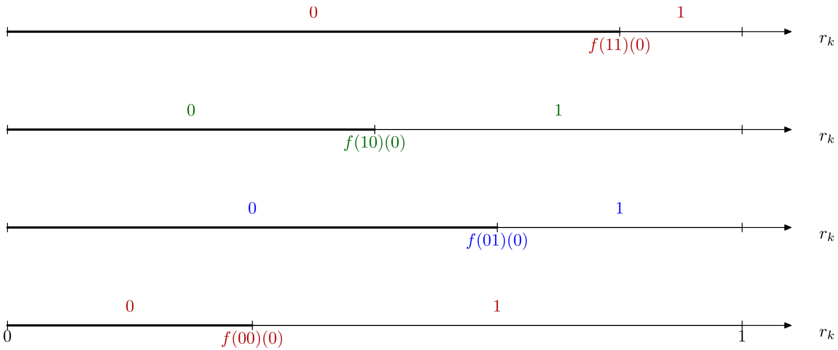
It defines an update function for F, given by:

$$\phi : \mathcal{A}^{\mathbb{Z}} \times [0, 1]^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$$

$$\phi(x, r)_k = \begin{cases} \mathbf{0} \text{ if } r_k < f((x_i)_{i \in k+\mathcal{N}})(0) \\ \mathbf{1} \text{ otherwise.} \end{cases}$$

Update function of a PCA

Example: $\mathcal{A} = \{0, 1\}$, neighbourhood $\mathcal{N} = \{0, 1\}$



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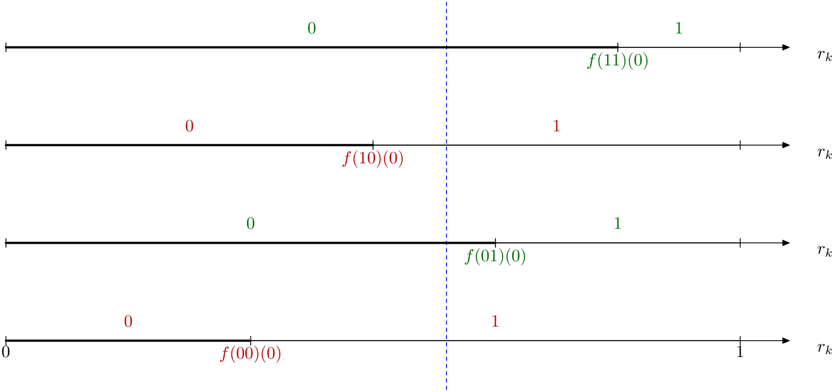

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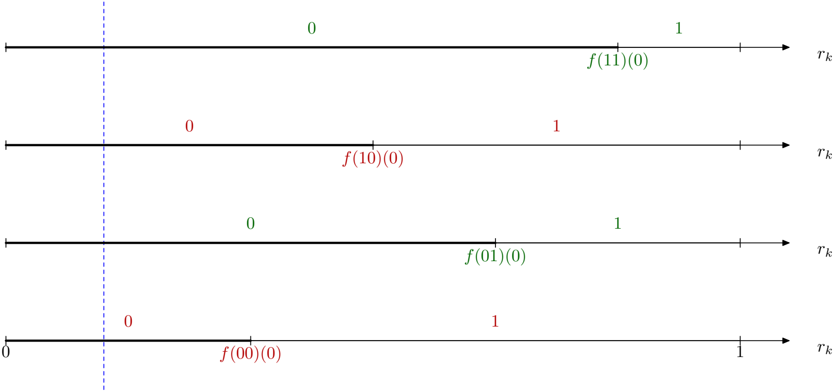

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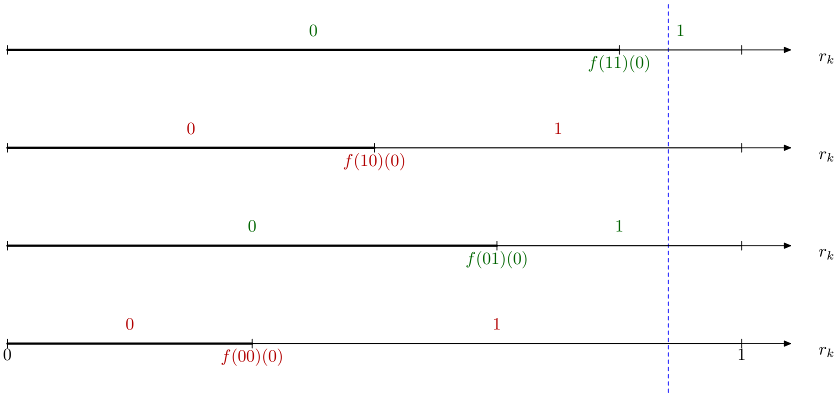
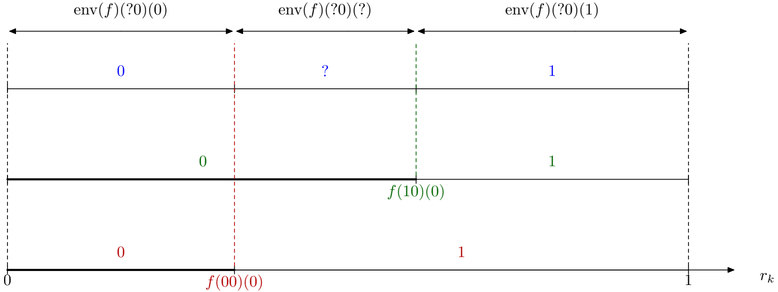


Envelope PCA

Introduction of an envelope PCA defined on the alphabet

$$\mathcal{B} = \{0 = \{0\}, 1 = \{1\}, ? = \{0, 1\}\},$$

to handle configurations partially known.



The update function $\tilde{\phi}$ of env(P) satisfies for $x \in \mathcal{A}^E$ and $y \in \mathcal{B}^E$,

$$x \in y \Rightarrow \forall r \in [0,1]^E, \phi(x, r) \in \tilde{\phi}(y, r).$$

Coupling from the past algorithm

Let F be an ergodic PCA on $E = \mathbb{Z}$, $\mathcal{A} = \{0, 1\}$, with $\mathcal{N} = \{0, 1\}$.

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$a \quad b$

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? ?

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$$\begin{array}{ccc} ? & ? & \\ ? & ? & ? \end{array} \qquad (r_i^1)_{0 \leq i \leq 2}$$

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?	1		
?	?	?	
?	?	?	?

$(r_i^1)_{0\leq i\leq 2}$
$(r_i^2)_{0\leq i\leq 3}$

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?	1			
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?	1				
?	0	1			$(r_i^1)_{0 \leq i \leq 2}$
?	?	?	?		$(r_i^2)_{0 \leq i \leq 3}$
?	?	?	?	?	$(r_i^3)_{0 \leq i \leq 4}$

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?	1				
?	0	1			$(r_i^1)_{0 \le i \le 2}$
1	?	?	0		$(r_i^2)_{0 \le i \le 3}$
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?	1					
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?	?	?	?	?		$(r_i^3)_{0 \le i \le 4}$
?	?	?	?	?	?	$(r_i^4)_{0 \le i \le 5}$

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?	1					
?	0	1				$(r_i^1)_{0 \le i \le 2}$
1	?	?	0			$(r_i^2)_{0 \le i \le 3}$
?	0	?	1	?		$(r_i^3)_{0 \le i \le 4}$
?	?	?	?	?	?	$(r_i^4)_{0 \le i \le 5}$

Coupling from the past algorithm

Let F be an ergodic PCA on $E = \mathbb{Z}$, $\mathcal{A} = \{0, 1\}$, with $\mathcal{N} = \{0, 1\}$.

?	1					
?	0	1				$(r_i^1)_{0 \leq i \leq 2}$
1	1	?	0			$(r_i^2)_{0 \leq i \leq 3}$
?	0	?	1	?		$(r_i^3)_{0 \leq i \leq 4}$
?	?	?	?	?	?	$(r_i^4)_{0 \leq i \leq 5}$

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0	**1**					
1	0	1				$(r_i^1)_{0 \le i \le 2}$
1	1	?	0			$(r_i^2)_{0 \le i \le 3}$
?	0	?	1	?		$(r_i^3)_{0 \le i \le 4}$
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Coupling from the past algorithm

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0	**1**					
1	0	1				$(r_i^1)_{0 \leq i \leq 2}$
1	1	?	0			$(r_i^2)_{0 \leq i \leq 3}$
?	0	?	1	?		$(r_i^3)_{0 \leq i \leq 4}$
?	?	?	?	?	?	$(r_i^4)_{0 \leq i \leq 5}$

Proposition

If this algorithm stops a.s. then it samples perfectly π.

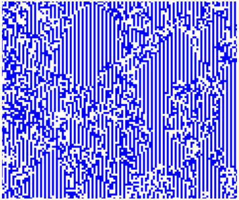
Perfect sampling using EPCA

Proposition
[Bušić-Mairesse-M., STACS 2011 & Adv. in Appl. Probab.]

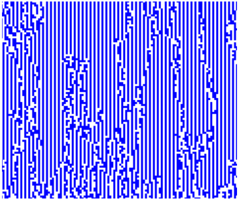
The algorithm stops a.s. if and only if the EPCA is ergodic.

- If the set of cells is finite, it is the case for positive-rate PCA.
- For $\mathbb{Z}^d$, there exists $\alpha^* \in]0,1[$ such that the EPCA is
  - ergodic if $\text{env}(f)(?^{\mathcal{N}})(?) < \alpha^*$,
  - non-ergodic if $\min_{x \in \mathcal{B}^{\mathcal{N}} \setminus \mathcal{A}^{\mathcal{N}}} \text{env}(f)(x)(?) > \alpha^*$.

The majority-flip PCA

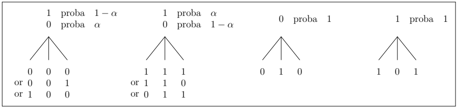


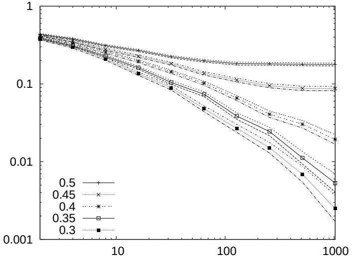
$\alpha = 0.5$



$\alpha = 0.3$

Study of the majority-flip PCA





Plan

1. Ergodicity and perfect sampling
   - Undecidability of the ergodicity
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   - Bernoulli invariant measures
   - Density classification
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Back to elementary PCA

$E = \mathbb{Z}$, $\mathcal{A} = \{0, 1\}$, neighbourhood of size 2.

$$\begin{array}{ll}
\begin{array}{lll} 1 & \text{with probability} & \theta_{11} \\ 0 & \text{with probability} & 1-\theta_{11} \\ \uparrow & & \\ 0\ \ 0 & & \end{array}
&
\begin{array}{lll} 1 & \text{with probability} & \theta_{01} \\ 0 & \text{with probability} & 1-\theta_{01} \\ \uparrow & & \\ 0\ \ 1 & & \end{array}
\\[2em]
\begin{array}{lll} 1 & \text{with probability} & \theta_{10} \\ 0 & \text{with probability} & 1-\theta_{10} \\ \uparrow & & \\ 1\ \ 0 & & \end{array}
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\end{array}$$

Bernoulli invariant measures

Proposition [Belyaev and al. 1969]

The Bernoulli measure $\mu_p = \mathcal{B}(p)^{\otimes \mathbb{Z}}$ is an invariant measure of the PCA iff its transitions probabilities satisfy (at least) one of the following equalities.

1. $(1-p) \cdot \theta_{00} + p \cdot \theta_{01} = (1-p) \cdot \theta_{10} + p \cdot \theta_{11} = p$
2. $(1-p) \cdot \theta_{00} + p \cdot \theta_{10} = (1-p) \cdot \theta_{01} + p \cdot \theta_{11} = p$

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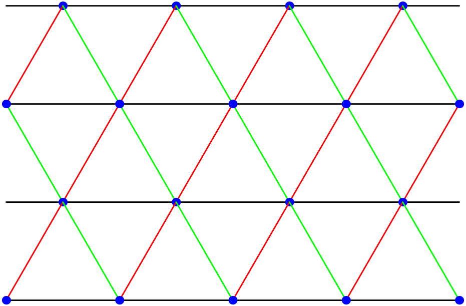
Each of these conditions implies surprising properties of the space-time diagrams.

Proposition [Mairesse-M., Ann. Inst. H. Poincaré Probab. Statist.]

When both conditions are satisfied,

- all the lines of the space-time diagram are i.i.d.



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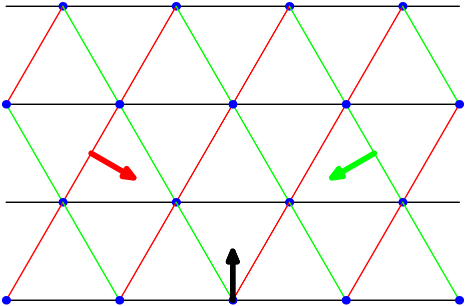
When both conditions are satisfied,

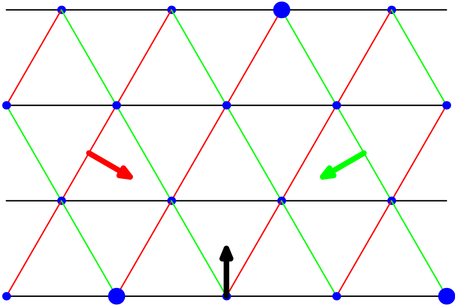
- all the lines of the space-time diagram are i.i.d.
- the PCA appears in three directions



Proposition [Mairesse-M., Ann. Inst. H. Poincaré Probab. Statist.]

When both conditions are satisfied,

- all the lines of the space-time diagram are i.i.d.
- the PCA appears in three directions
- three points are independent unless they form an equilateral triangle pointing up.

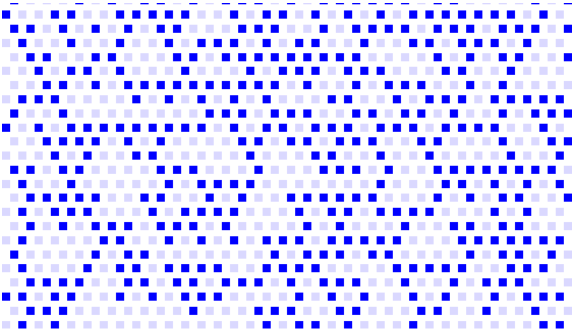


Example 1

The choice $\theta_{01} = \theta_{10} = s$ and $\theta_{00} = \theta_{11} = 1 - s$ corresponds to

$$f(x, y) = s \cdot \delta_{x+y \mod 2} + (1 - s) \cdot \delta_{x+y+1 \mod 2}.$$

The measure $\mu_{1/2}$ is invariant.



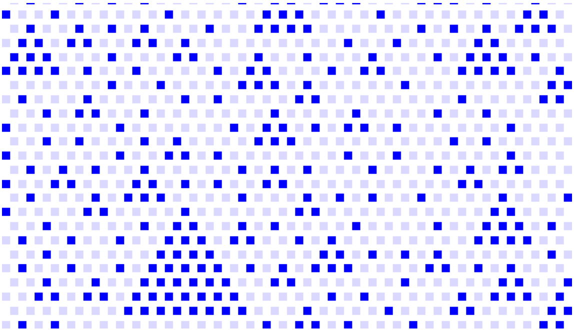
(s=3/4)

Example 2

For every $p \in [0, 1/2]$, one can set

$$\theta_{01} = \theta_{10} = 0, \quad \theta_{11} = 1 \text{ and } \theta_{00} = p/(1-p).$$

This PCA forbids elementary triangles (pointing up) having a single 0.



(p=1/3)

Larger alphabet

Alphabet $\mathcal{A} = \{0, \ldots, n\}$.
$\theta_{ij}^k =$ probability to get k if the neighbourhood is in state ij.

Proposition [Mairesse-M., Ann. Inst. H. Poincaré Probab. Statist.]

The Bernoulli measure $\mu_p \quad (p = (p_0, \ldots, p_n))$ is invariant if one of the following conditions is satisfied.

$$\forall i \in \mathcal{A}, \forall k \in \mathcal{A}, \sum_{j \in \mathcal{A}} p_j \, \theta_{ij}^k = p_k$$

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Same properties of space-time diagrams.

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Same properties of space-time diagrams.

Remark: the deterministic CA that we recover are permutative CA.

The density classification problem

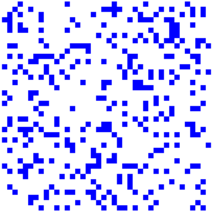
The density classification problem

$E = \mathbb{Z}^d$, $\mathcal{A} = \{0, 1\}$.
We still denote by μ_p the Bernoulli measure of parameter p.

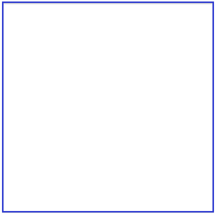
Challenge

The *density classification problem* consists in finding a (P)CA or an IPS F, such that:

$$\begin{cases} p < 1/2 \implies \mu_p F^t \xrightarrow[t\to\infty]{w} \delta_{\underline{0}}, \\ p > 1/2 \implies \mu_p F^t \xrightarrow[t\to\infty]{w} \delta_{\underline{1}}. \end{cases}$$

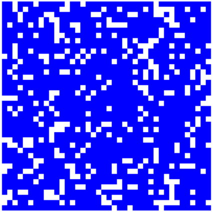


$\xrightarrow[t\to\infty]{w}$

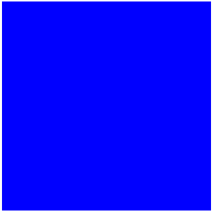


p=0.2

p=0



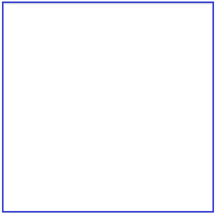
$\xrightarrow[t\to\infty]{w}$



p=0.8

p=1

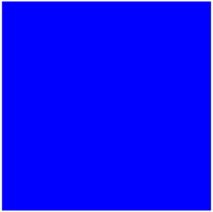
$\xrightarrow[t\to\infty]{w}$



p=0.49

p=0

$\xrightarrow[t\to\infty]{w}$



p=0.51

p=1

A solution on $\mathbb{Z}^2$

Definition of Toom's CA

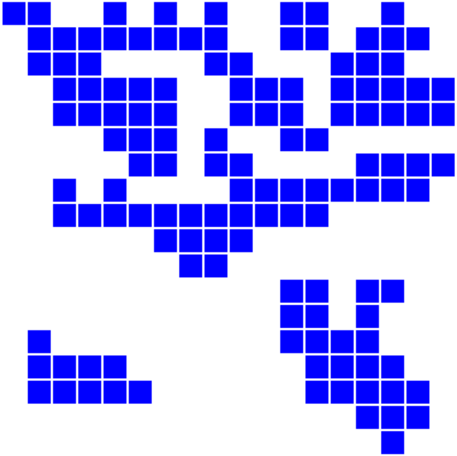
Toom's CA is the CA $\mathcal{T}$ on $\mathbb{Z}^2$ of neighborhood $\mathcal{N} = \{(0,0),(0,1),(1,0)\}$ (north-east-center) defined by the majority rule, that is,

$$(\mathcal{T}(x))_{i,j} = \mathrm{maj}(x_{i,j}, x_{i,j+1}, x_{i+1,j}).$$

Proposition [Bušić-Fatès-Mairesse-M., LATIN 2012 & Electron. J. Probab. 2013]

Toom's rule classifies the density.

The proof in pictures



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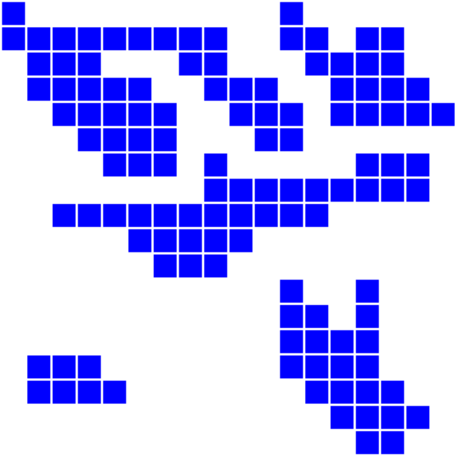

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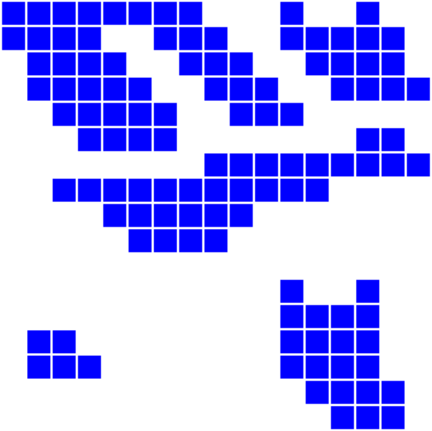

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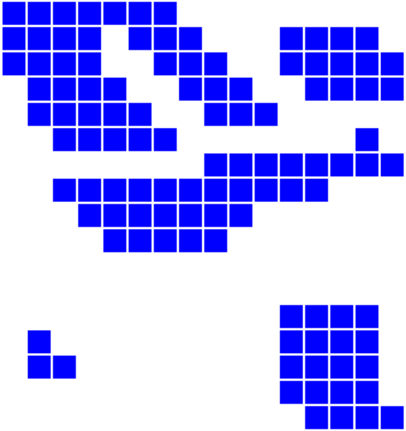

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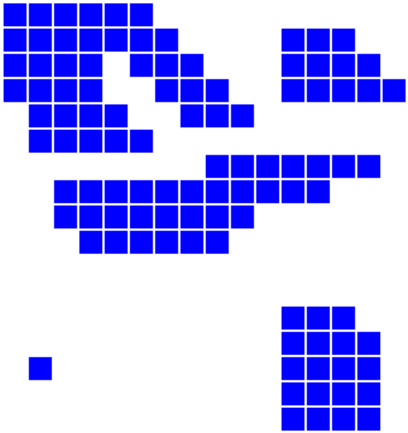

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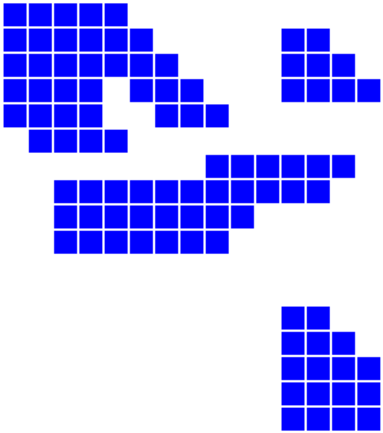

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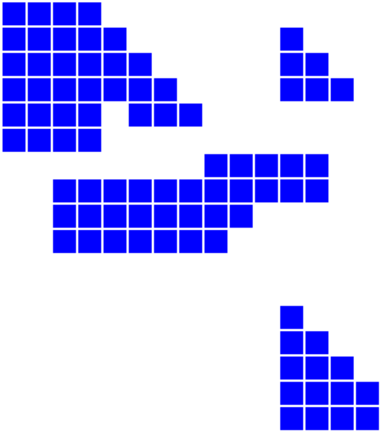

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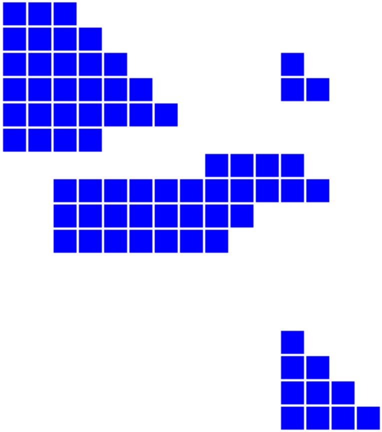

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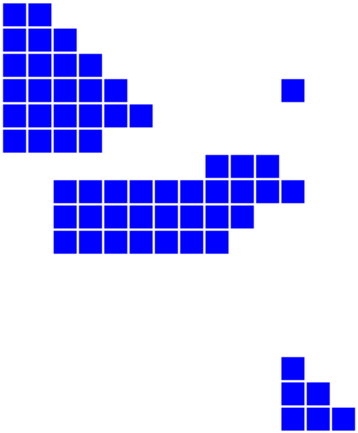

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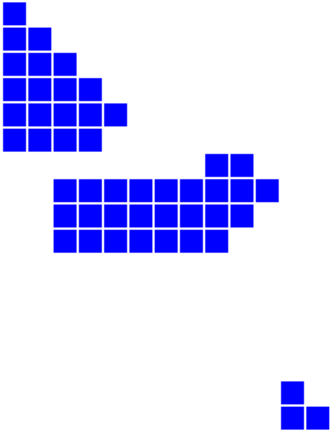

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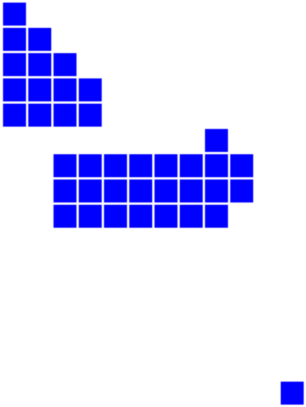

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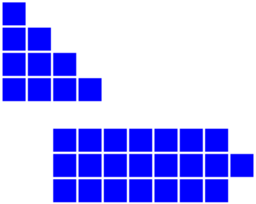

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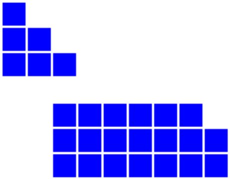

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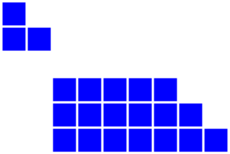

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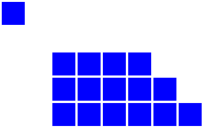

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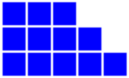

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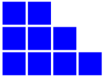

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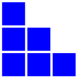

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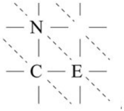

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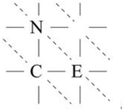

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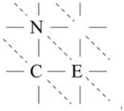

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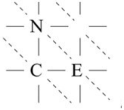

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- Any finite 1-cluster disappears in finite time and always stays in its enveloping rectangle
- A given point belongs a.s. to the enveloping rectangle of an at most finite number of 1-clusters (by the exponential decay of the size of 1-clusters)

Summary

	$\mathbb{Z}/n\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^d, d \geq 2$	Tree $T_n, n \geq 3$
CA	No perfect solution, good performances of GKL (huge literature)		Toom's rule No symmetric majority	Asymmetric majority
PCA	Arbitrary good precision with Majority-traffic No perfect solution		Toom's rule	Asymmetric majority
IPS	No perfect solution		Modified Toom's rule	Asymmetric majority

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Link with the positive rate problem.

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Motivation: understanding the combinatorics of multi-dimensional SFT, being able to generate patterns “uniformly”.

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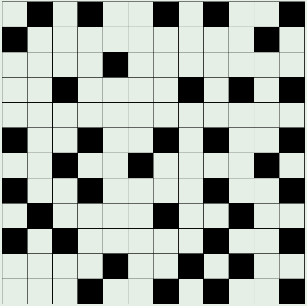
Example: two-dimensional Fibonacci SFT

Set of configurations without two consecutive black squares, vertically or horizontally.



One-dimensional subshift of finite type

Let $\mathcal{A}$ be an alphabet with n letters, and let $A \in \mathcal{M}_n(\{0,1\})$.

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Subshift of finite type

The *subshift of finite type* associated to A is the set Σ_A of words $w \in \mathcal{A}^{\mathbb{Z}}$ such that if $A_{i,j} = 0$, w does not contain the pattern ij.

$$A_{i,j} = \begin{cases} 1 \text{ if } ij \text{ is an allowed pattern,} \\ 0 \text{ if } ij \text{ is a forbidden pattern.} \end{cases}$$

$$\Sigma_A = \{w \in \mathcal{A}^{\mathbb{Z}}; \forall k \in \mathbb{Z}, A_{w_k, w_{k+1}} = 1\}.$$

The Parry measure

The Parry measure

Let λ be the Perron value of the matrix A (assumed to be irreducible and aperiodic), and let r be the right-eigenvector associated to λ, satisfying $\sum_{i=1}^{n} r_i = 1$. The *Parry measure* is the Markov measure π of transition matrix P defined, for any $i, j \in \mathcal{A}$, by

$$P_{i,j} = A_{i,j} \frac{r_j}{\lambda r_i}.$$

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The Parry measure is Markov-uniform: for a given $k \geq 1$, the value $\pi(awb)$ does not depend of the word $w \in \{1, \ldots, n\}^k$ such that awb is allowed.

Measure of maximal entropy

Theorem

Let $\mathcal{M}_{\Sigma_A}$ be the set of translation invariant measures on the SFT Σ_A, and let $\pi \in \mathcal{M}_{\Sigma_A}$. The following properties are equivalent.

(i) π is the Parry measure associated to Σ_A,

(ii) π is a Markov-uniform measure on Σ_A,

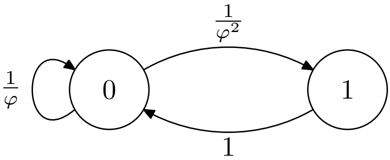
(iii) π is the measure of maximal entropy of Σ_A.

Example: Fibonacci SFT

Let $\mathcal{A} = \{0, 1\}$. The one-dimensional Fibonacci SFT is the set of words that do not contain two consecutive 1's. It is given by:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Its Parry measure is the Markov measure given by



with $\pi_0 = \frac{\varphi^2}{1+\varphi^2}$ and $\pi_1 = \frac{1}{1+\varphi^2}$.

First way to generate the Parry measure

The Parry measure of Fibonacci SFT can be generated by:

- choosing independently to write a 0 with probability $r_0 = \frac{1}{\varphi}$ and a 1 with probability $r_1 = \frac{1}{\varphi^2}$,
- rejecting the 1's creating forbidden patterns.

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Lemma [M. 2013]

For any SFT, the Parry measure can be generated by independent draws of letters, with reject of a letter if it creates a forbidden pattern.

Second way to generate the Parry measure

The Parry measure of Fibonacci SFT can be generated by:

- choosing independently to write a 0 with probability $\tilde{r}_0 = \frac{1}{\varphi^2}$ and a 1 with probability $\tilde{r}_1 = \frac{1}{\varphi}$,
- deleting pairs of consecutive 1's.

Second way to generate the Parry measure

The Parry measure of Fibonacci SFT can be generated by:

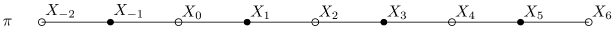
- choosing independently to write a 0 with probability $\tilde{r}_0 = \frac{1}{\varphi^2}$ and a 1 with probability $\tilde{r}_1 = \frac{1}{\varphi}$,
- deleting pairs of consecutive 1's.

Proposition [M. 2013]

For *confluent SFT*, the Parry measure can be generated by independent draws of letters and deletion of forbidden patterns.

Link with PCA

Consider a configuration distributed according to the Parry measure π of the Fibonacci SFT.



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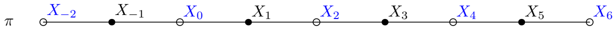
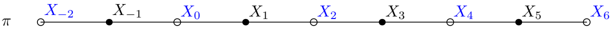


For all $i \in \mathbb{Z}$, if $X_{2i} = X_{2i+2} = 0$, we flip the value of X_{2i+1} with probability $1/2$.
By the Markov-uniform property, the new sequence is still distributed according to π.

Link with PCA

Consider a configuration distributed according to the Parry measure π of the Fibonacci SFT.



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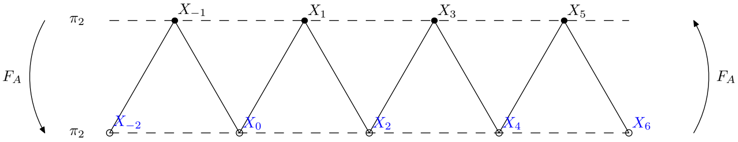
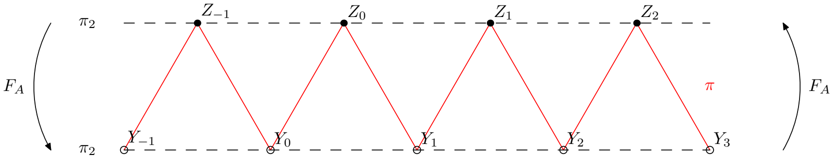


Link with PCA

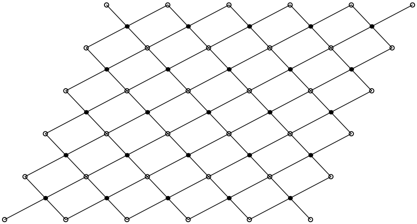
$$\begin{array}{c} 1 \text{ with probability } 1/2 \\ 0 \text{ with probability } 1/2 \\ \uparrow \\ 0 \quad 0 \end{array} \qquad\qquad \begin{array}{c} 0 \text{ (with probability } 1) \\ \uparrow \\ 0 \quad 1 \\ \text{or } 1 \quad 0 \\ \text{or } 1 \quad 1 \end{array}$$

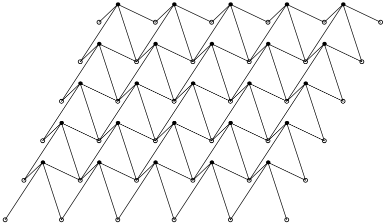


The projection π_2 of the Parry measure on odd (resp. even) sites is an invariant measure of the PCA.

Extension

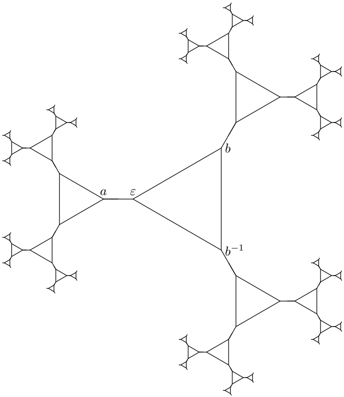
The analogous result holds for any SFT in dimension 1 or in higher dimension.



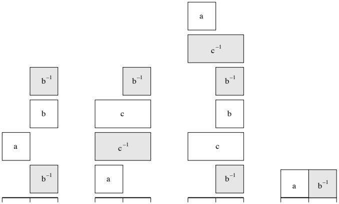


Perspectives and works in progress

- Random walks on free products of groups (with J. Mairesse)



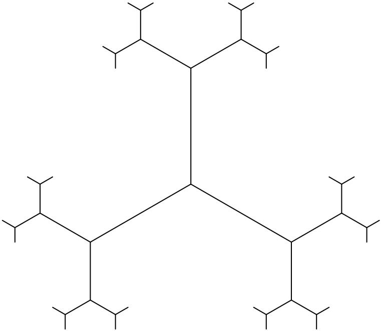
$\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$



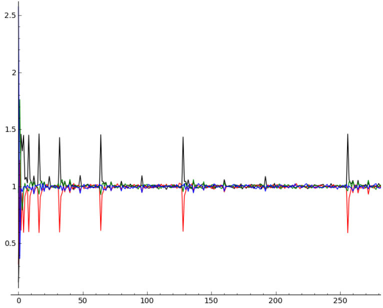
$\mathbb{Z}^2 * \mathbb{Z}$

Study of the limit measure

- Random walks on free products of groups (with J. Mairesse)
- Generalisation of the Parry measure to SFT defined on infinite trees (with V. Delecroix)

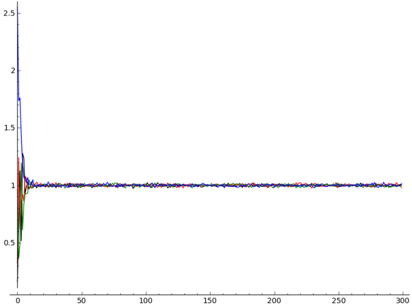


- Random walks on free products of groups (with J. Mairesse)
- Generalisation of the Parry measure to SFT defined on infinite trees (with V. Delecroix)
- Deterministic CA and Bernoulli invariant measures
  Rigidity and randomisation (with B. Hellouin de Menibus, A. Maass, M. Sablik)



$f(x, y) = x + y$

$\mathcal{A} = \mathbb{Z}/4\mathbb{Z}$



$f(x + y) = \tau_{23}(x + y)$

- Random walks on free products of groups (with J. Mairesse)
- Generalisation of the Parry measure to SFT defined on infinite trees (with V. Delecroix)
- Deterministic CA and Bernoulli invariant measures
  Rigidity and randomisation (with B. Hellouin de Menibus, A. Maass, M. Sablik)
- Ergodicity of this PCA for large values of p. Work related to the study of a "percolation game" (with J. Martin)