# Kinetically constrained models and bootstrap percolation

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  - link with boostrap percolation cellular automaton
  - relaxation times are finite at any density $\rho \in [0,1)$
  - sharp divergence when $\rho \to 1$
- Spiral model (G.Biroli, D.Fisher, C.T. '06):
  - dynamical glass transition at $\rho_c$, $\rho_c \in (0,1)$
  - relaxation times diverge at $\rho_c$
  - a finite fraction of particles is frozen at and above $\rho_c$
  - $\leftrightarrow$ discontinuous/critical percolation transition
    for a proper cellular automaton.

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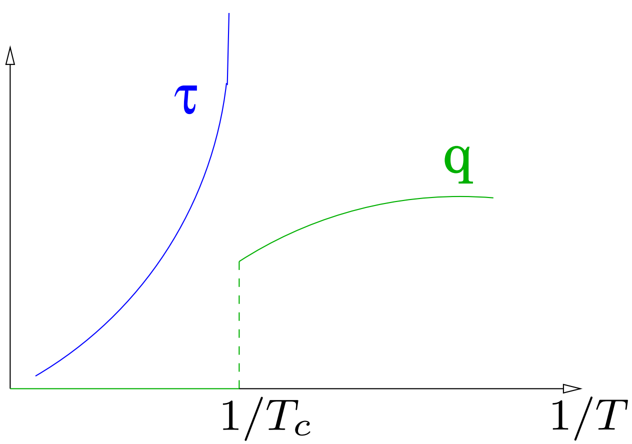


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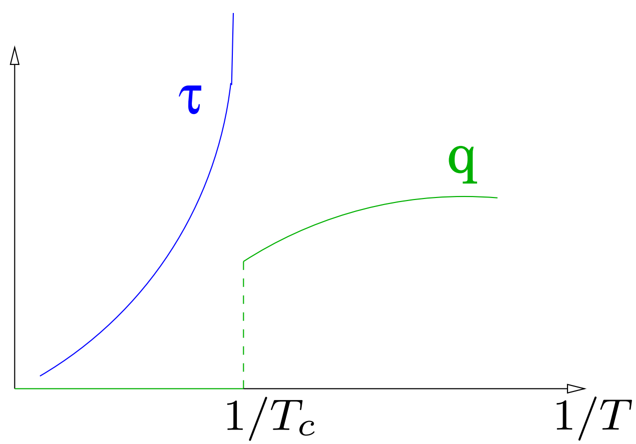


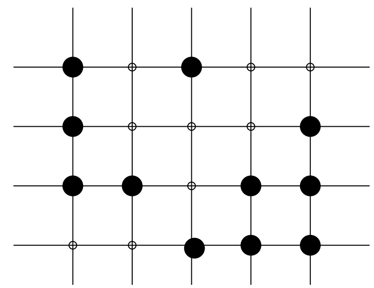

Ideal glass transition

KCM: looking for models with an ideal glass transition...

# FA model (d=2)

Interacting particle system with Glauber (birth/death) dyn.

Configurations: $\eta \in \{0,1\}^{|\mathbb{Z}^2|}$



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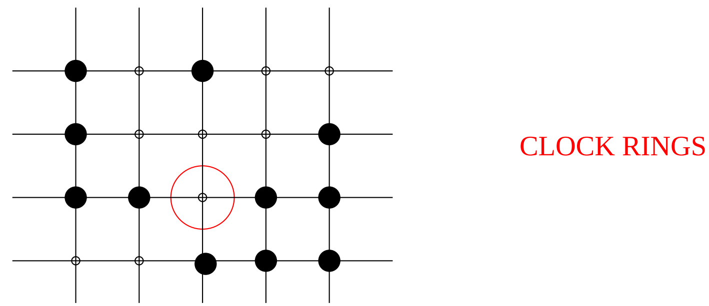


Kinetic constraint:

update only if at least two empty nearest neighbours

Constraint not satisfied: nothing happens

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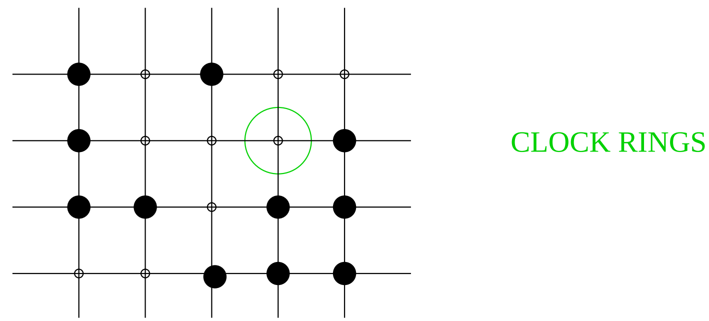


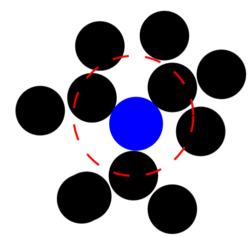

Kinetic constraint:
update only if at least two empty nearest neighbours

Constraint satisfied: update putting a particle with probability $\rho$ and an empty site with probability $1-\rho$

Namely, if constraint satisfied: $0 \xrightarrow{\rho} 1$ and $1 \xrightarrow{1-\rho} 0$

# Liquid at high density (low temp.)

- Cage effect: before moving a caged molecule its neighbours should move



size of cage increases with $\rho$ (neighbours are caged, neighbours of neighbours are caged,..)

- Idea: blocked structures percolate at $\rho_c < 1$
$\rightarrow$ key mechanism for glass transition
- FA constraint mimick local cage effect..
- Numerical simulations: $\rho \uparrow$ sharp slowing down
- Conjecture: ideal glass transition for FA at $\rho_c < 1$

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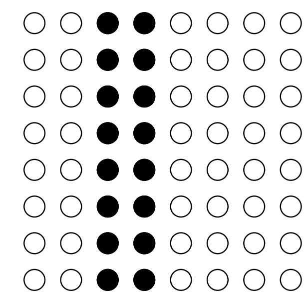

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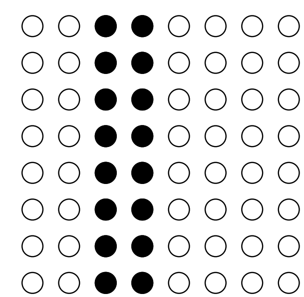

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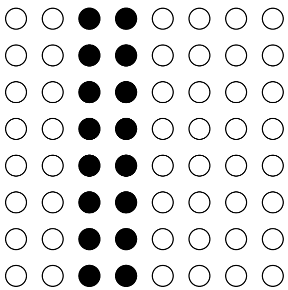

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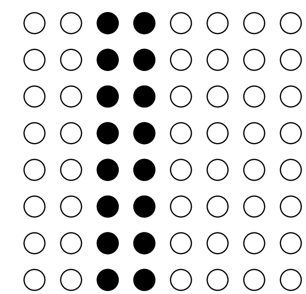

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- FA ('84) $\rho_c < 1$, Van Enter ('88) No! $\rho_c = 1$

# Does $\eta$ contain a blocked cluster?

## Deterministic discrete time cellular automaton

$$\eta_0 = \eta, \quad \eta_t = T\eta_{t-1}$$

$$T\eta(x) = \begin{cases} 0 & \text{if} \qquad\qquad\qquad\qquad \eta(x) = 0 \\ 0 & \text{if} \quad \eta(x) = 1 \text{ and } \sum_{y \; n.n. \; x}(1-\eta_y) \geq 2 \\ 1 & \text{if} \quad \eta(x) = 1 \text{ and } \sum_{y \; n.n. \; x}(1-\eta_y) < 2 \end{cases}$$

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2) Kill particles which have at least $2$ empty neighbours

Iterate until a stable configuration is reached...

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Iterate until a stable configuration is reached...
Particles of final config. = blocked particles of $\eta$ under FA
If final conf. is empty $\rightarrow \eta$ had no blocked cluster

# Algorithm = Bootstrap (empty $\leftrightarrow$ filled)

- No bootstrap percolation transition (Van Enter '87).
  $\rho_c = 1$ where

$$\rho_c := \sup\{\rho : \mu_\infty^\rho(\eta(0) = 1) = 0\}$$

  and $\mu_t^\rho$ is the evoluted of $\mu_\rho$ under $T^t$
- Higher spatial dimensions $d \geq 3$, kill particles if at least $f$ empty neighbours: no bootstrap percolation transition $\rho_c = 1\ \forall d, f \leq d$; $\rho_c = 0\ \forall d, f > d$ (Schonmann '92).
- $\rightarrow$ No blocked fraction of particles above a critical density for FA, no ideal glass transition!

Cancrini, Martinelli, Roberto, C.T. '08:

- spectral gap $> 0\ \ \forall\ \rho < 1 \rightarrow \tau < \infty$
- $\tau \geq \ell(\rho)$ with $\ell(\rho) = \exp\left(\frac{c}{1-\rho}\right)$ size of clusters which are unblocked under BP only sequentially from their border

# Spiral Model

Looking for constraints s.t. at $\rho_c < 1$ blocked clusters percolate and the density of blocked sites is $> 0$ at $\rho_c$ . . .

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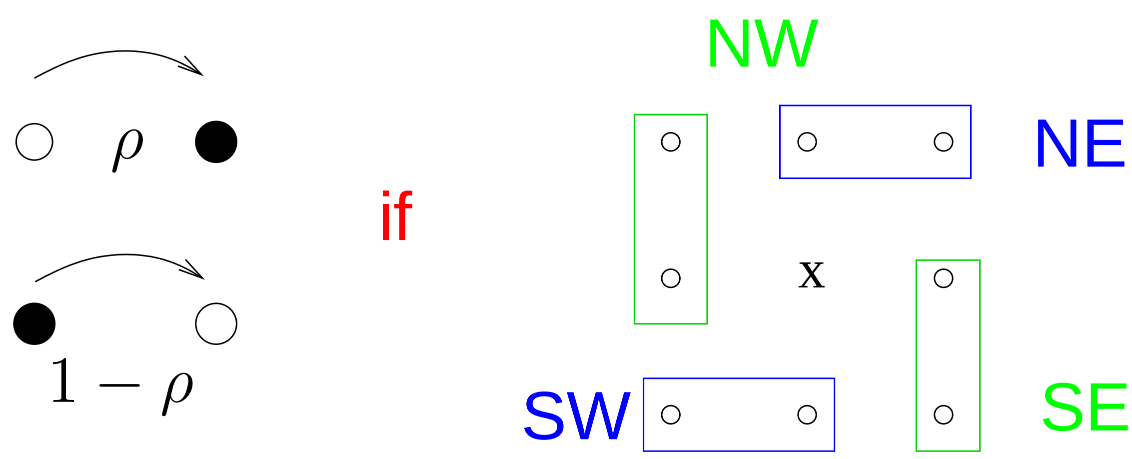
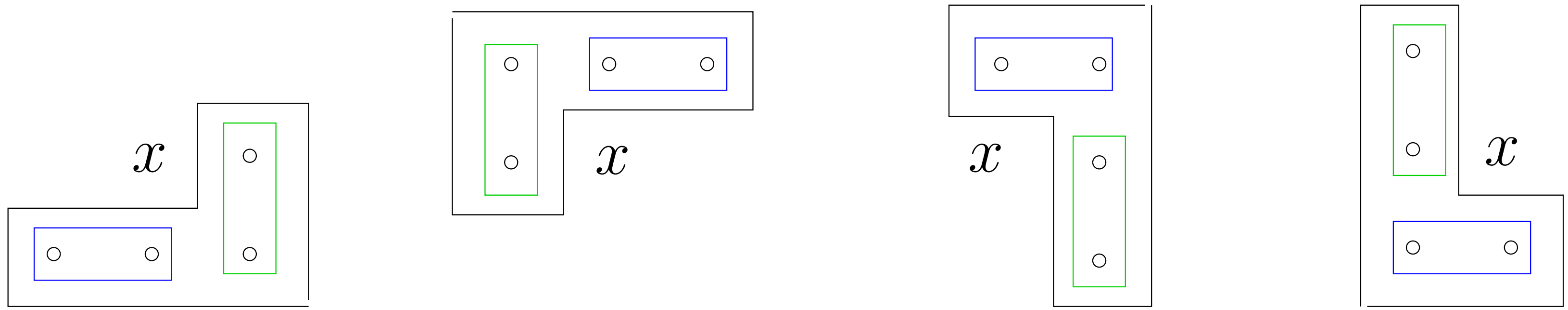
- Evolution: Glauber dynamics




(NE OR SW empty) AND (NW OR SE empty )

# Spiral constraint




At least one of these L-shaped half cages should be empty

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- $\leftrightarrow$ The bootstrap-like algorithm with spiral constraints has a discontinuous percolation transition at $\rho^{OP}$

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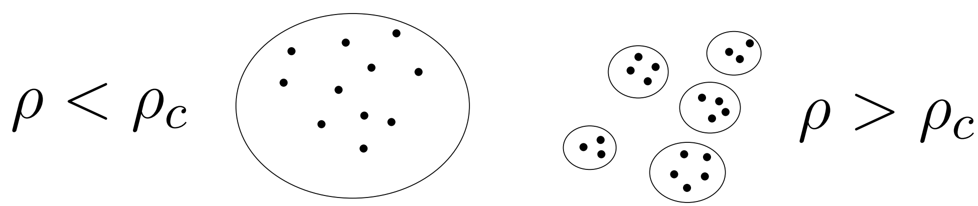


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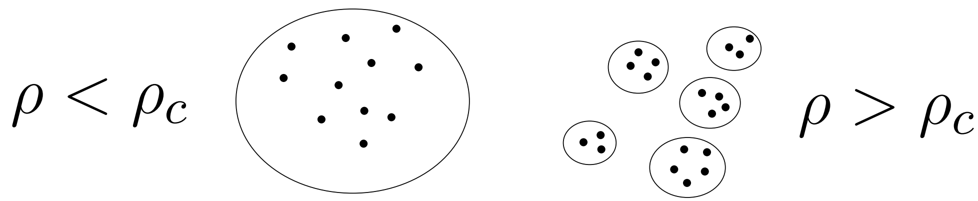


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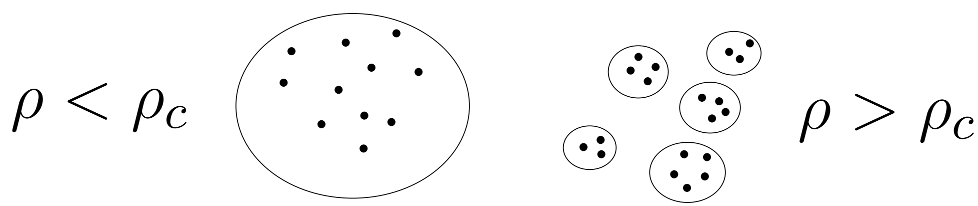


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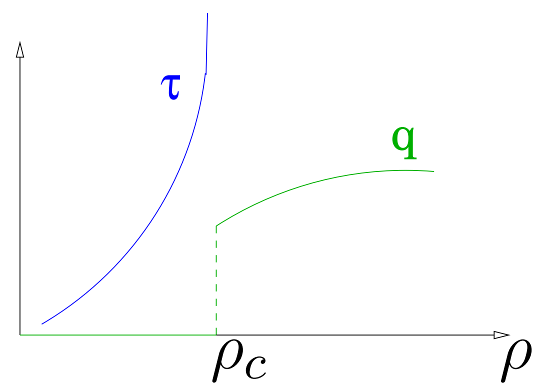


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Spiral model has an ideal glass transition

# Key ideas

Percolation transition for spiral cellular automaton: discontinuous + anomalous critical properties

Key ingredients:

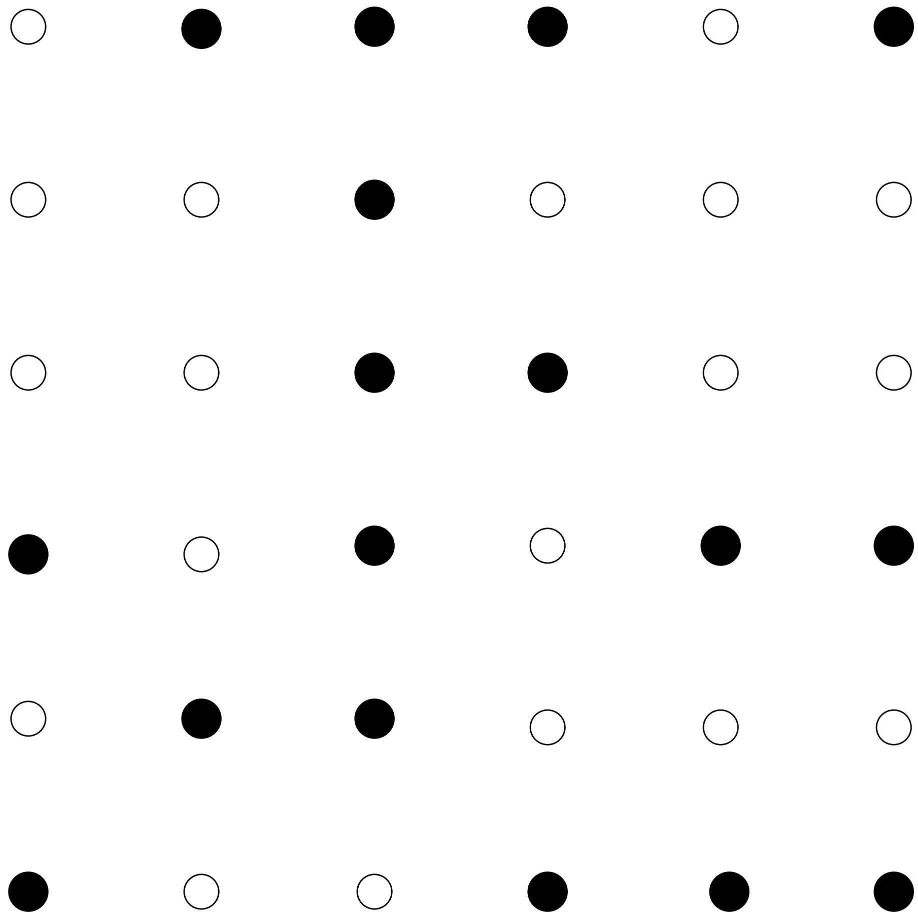
- two transverse blocking directions
- oriented percolation at $\rho_c$ in each direction

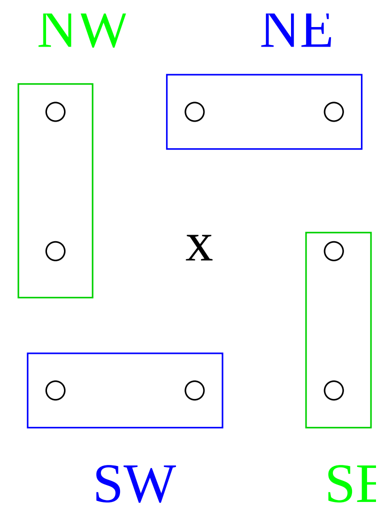
[?] Missing technical point: prove an anysotropic property for standard oriented percolation

- Cellular automata play a key role in understanding the behavior of KCM for glassy systems
- Some cellular automata devised to study KCM may display novel (interesting?) behavior

Thanks!

# $\rho > \rho^{OP}$: blocked clusters





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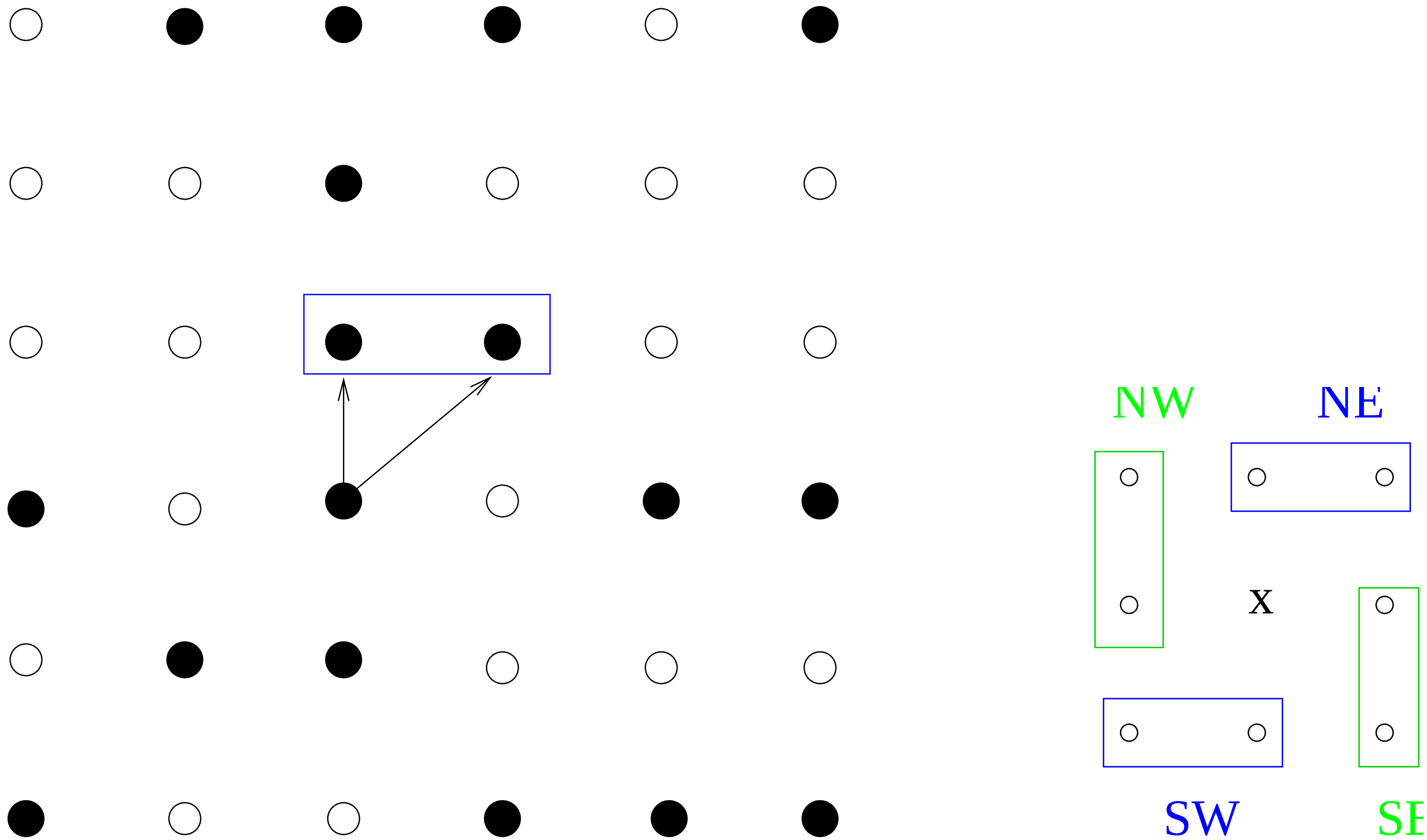


Arrows towards NE neighbours and from SW neighbours.

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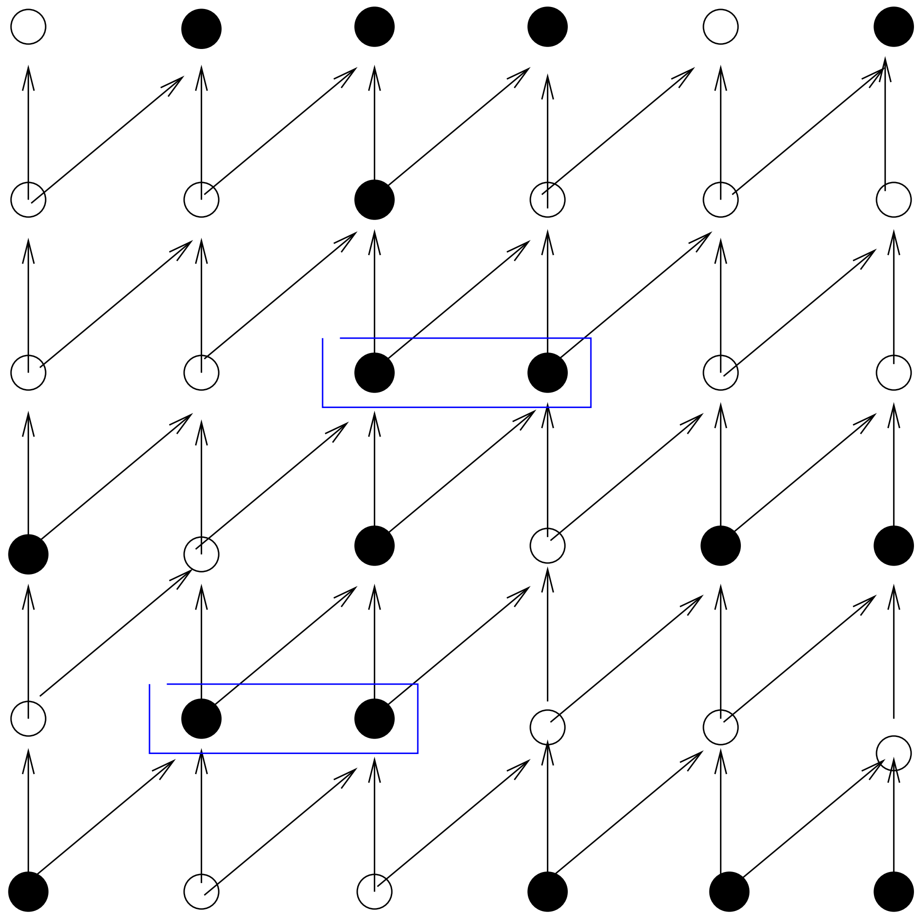

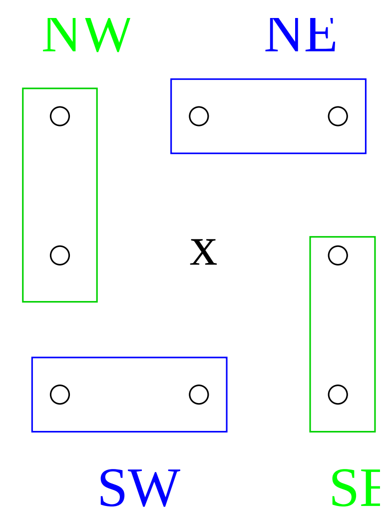


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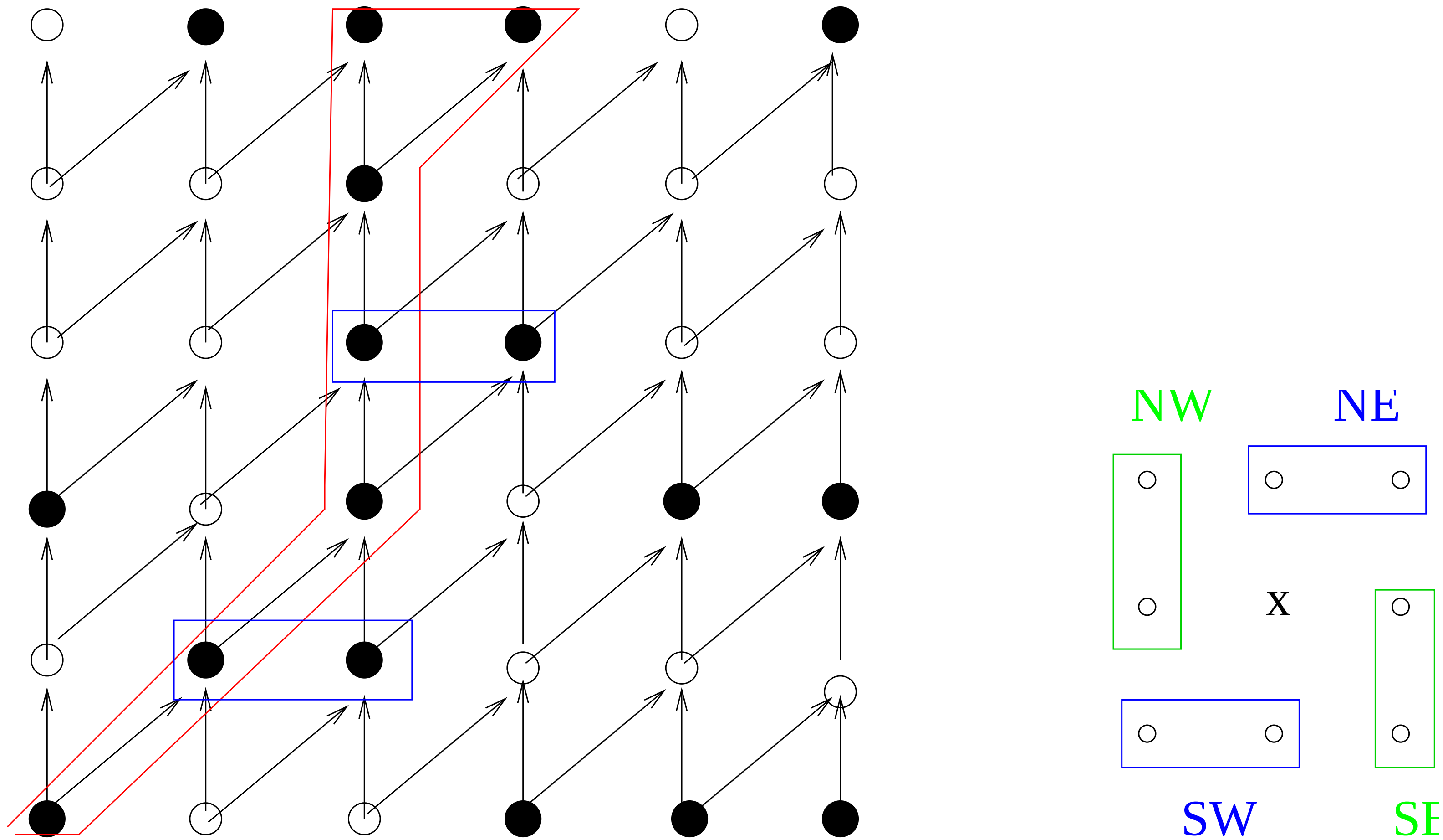
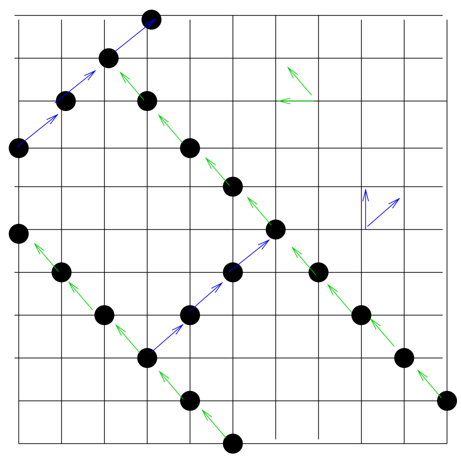



Arrows towards NE neighbours and from SW neighbours.
$\rightarrow$ NE-SW oriented clusters percolate

# Blocked clusters

Spanning NE-SW clusters $\longrightarrow$ blocked
Spanning NW-SE clusters $\longrightarrow$ blocked

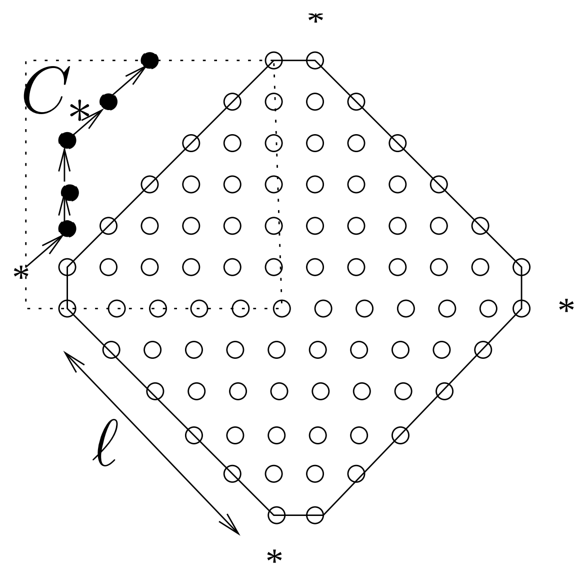
Blocked clusters $\not\longrightarrow$ spanning NE-SW or NW-SE clusters



We can construct blocked clusters by connecting finite NE-SW and SE-NW clusters

# $\rho < \rho^{OP}$: no blocked clusters

There exists large voids from which the bootstrap-like algorithm with spiral constraints empties the whole lattice




We can expand of one step if $|C_*| \leq \ell$

$P_\ell$:=Proba ($\ell \times \ell \ \rightarrow \ (\ell+2) \times (\ell+2)) \propto (1 - c\exp^{-c\ell})^4$

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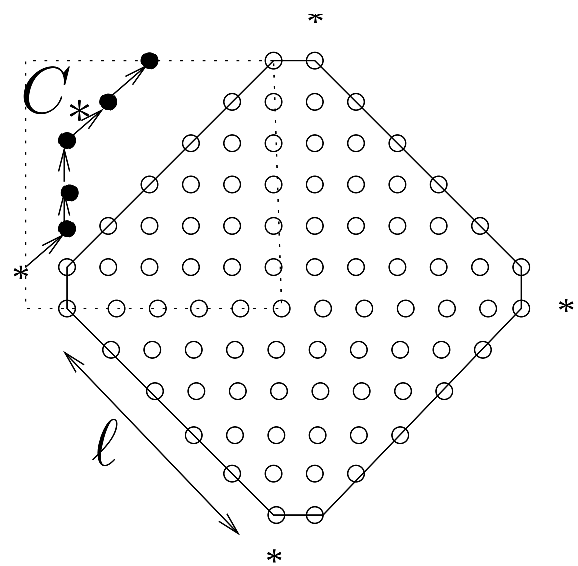


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$\rightarrow \prod_\ell^\infty P_\ell > 0 \rightarrow$ No blocked clusters $\rho < \rho^{OP}$

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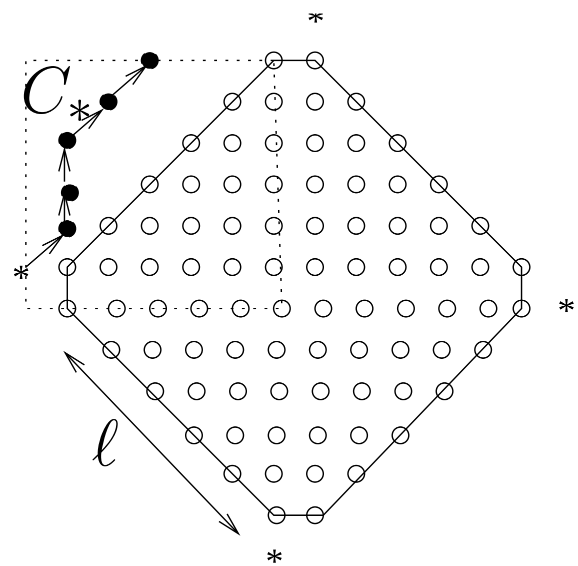


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$\to \prod_\ell^\infty P_\ell > 0 \to$ No blocked clusters $\rho < \rho^{OP}$

$\Rightarrow$ Percolation trans. at $\rho^{OP}$

# Discontinuity

$q(\rho_c) := \mu_{\rho_c}(\text{origin belongs to a blocked cluster}) > 0$

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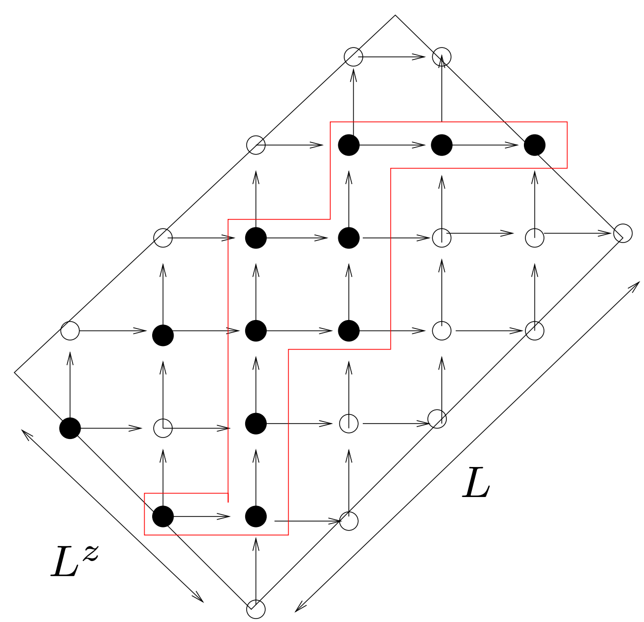
Key ingredients:

• NE-SW clusters with both ends on SE-NW clusters are blocked

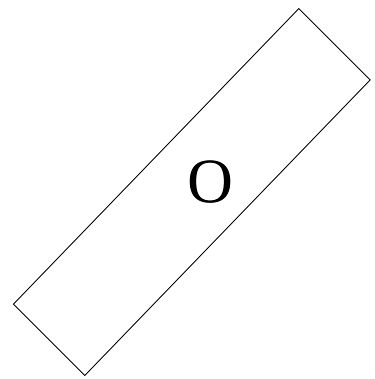
• anisotropy of OP $\Rightarrow$?

$\lim_{L\to\infty} \mu_{\rho_c}(\text{percolating cluster in } L \times L^z) > 0, \quad z < 1$



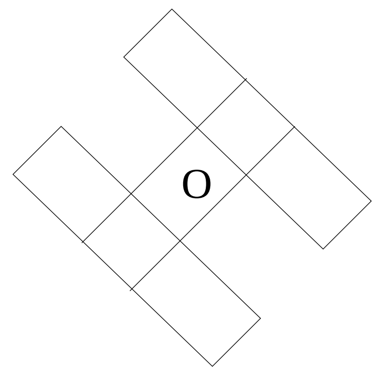
# Discontinuity

$q(\rho_c) := \mu_{\rho_c}(\text{origin belongs to a blocked cluster}) > 0$



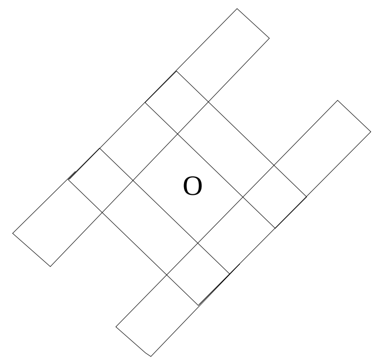
# Discontinuity

$q(\rho_c) := \mu_{\rho_c}(\text{origin belongs to a blocked cluster}) > 0$



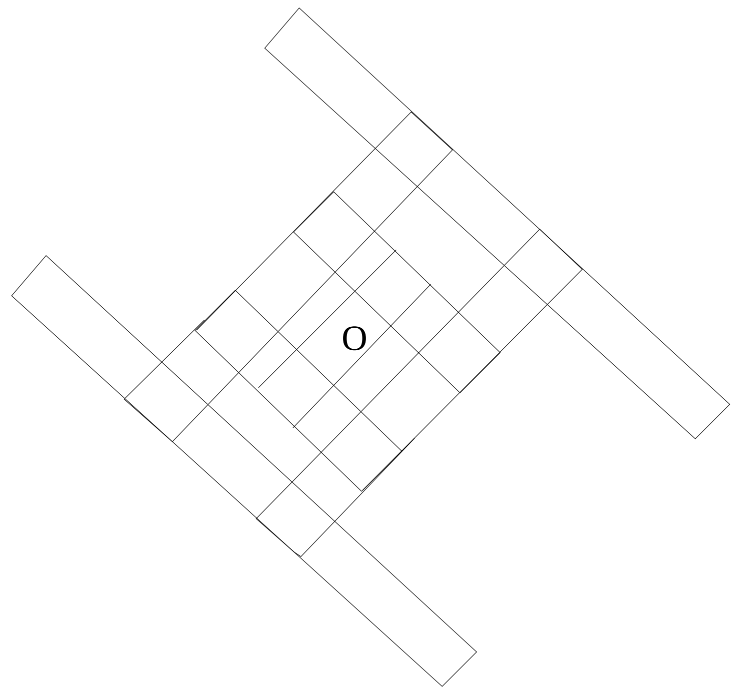
# Discontinuity

$q(\rho_c) := \mu_{\rho_c}($origin belongs to a blocked cluster$) > 0$



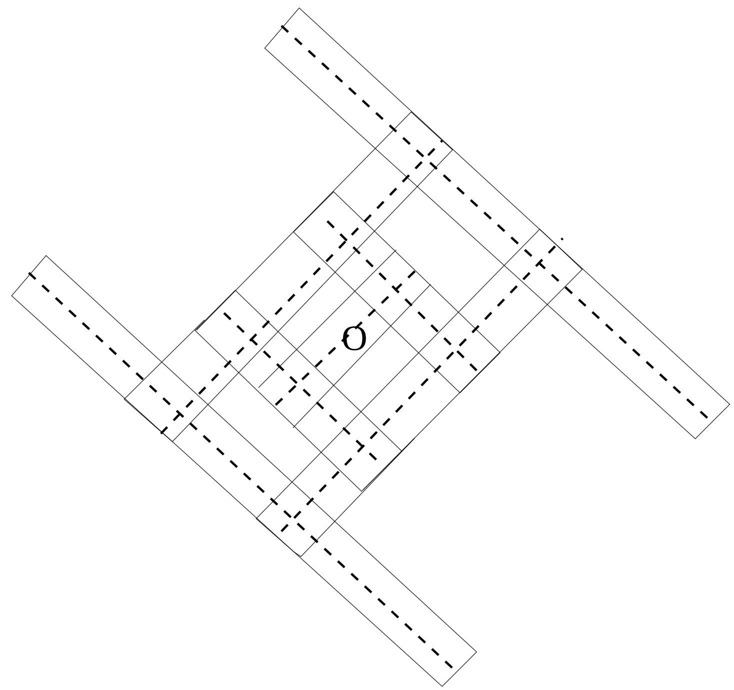
# Discontinuity

$q(\rho_c) := \mu_{\rho_c}($origin belongs to a blocked cluster$) > 0$



# Discontinuity

$q(\rho_c) := \mu_{\rho_c}($origin belongs to a blocked cluster$) > 0$



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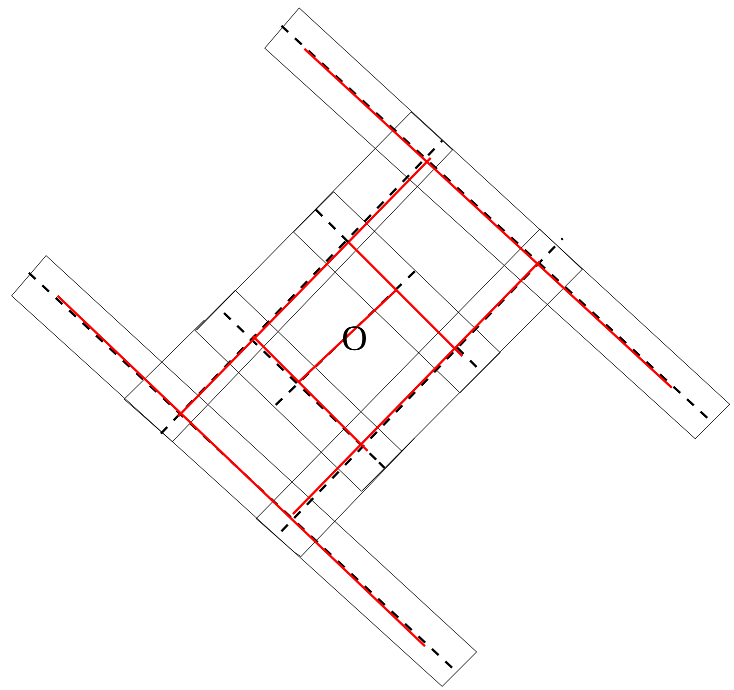
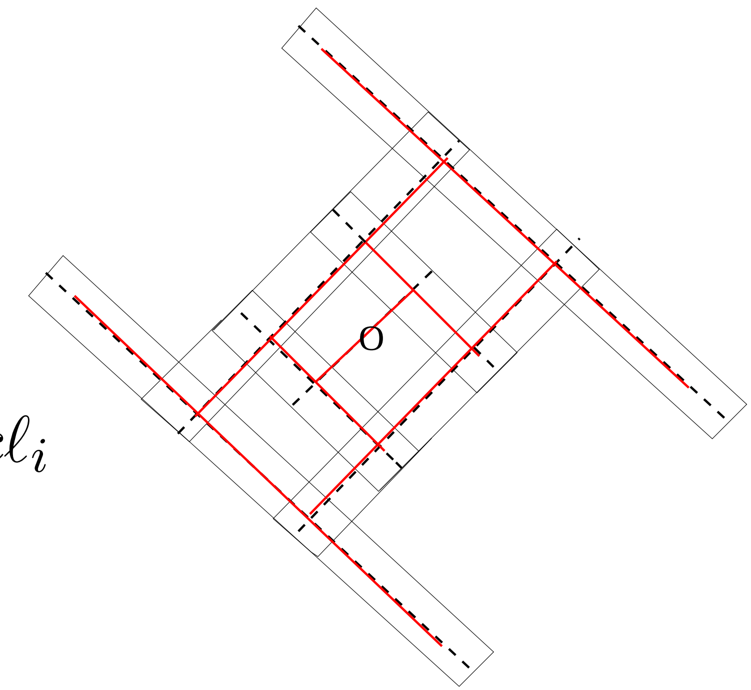


# Discontinuity

$q(\rho_c) := \mu_{\rho_c}($origin belongs to a blocked cluster$) > 0$




Rectangles $\ell_i \times c\ell_i$
with $\ell_{i+1} = 2\ell_i$

$$P_L := \mu_{\rho_c}(\exists \text{ cluster OP in } L \times cL) > 1 - \exp(-\alpha L^{1-z})$$

$$\rightarrow q(\rho_c) \geq c \prod_{i=1}^{\infty} (P_{\ell_i})^4 > 0$$

Clusters are compact at $\rho_c \rightarrow$ discontinuous percolation