Decidability and Universality in Stochastic Cellula Automata

Rencontres ACP

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Overview of the talk

1. Definitions

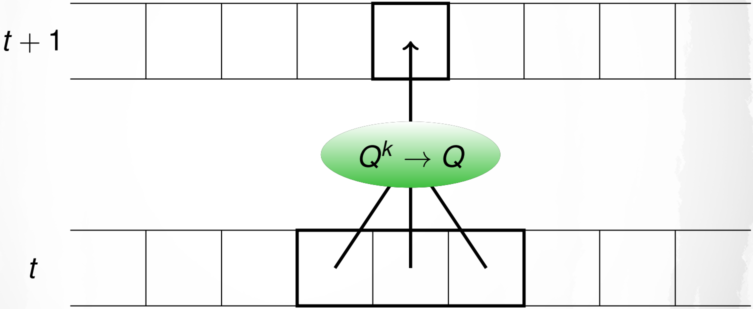
2. Decision problems

3. Simulations

Overview of the talk

Deterministic CA

- states Q, k neighbors

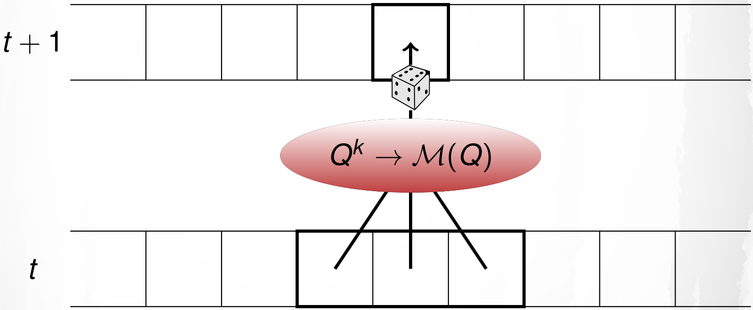


Hedlund's Theorem

Locally defined $\Leftrightarrow$ continuous shift-commuting

Correlation-free probabilistic CA

- states Q, k neighbors



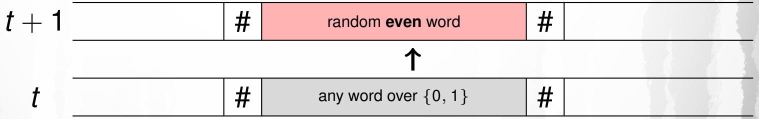
- global function: $Q^{\mathbb{Z}} \to \mathcal{M}(Q^{\mathbb{Z}})$ then $\mathcal{M}(Q^{\mathbb{Z}}) \to \mathcal{M}(Q^{\mathbb{Z}})$
- **examples:** noisy CA, α-asynchronous CA

Limitations

Arrighi-Fargetton-Nesme-Thierry (CiE'11)

Applying Causality Principles to the Axiomatization of Probabilistic Cellular Automata

1. no stability by iteration
   - image by F of uniform conf. is a Bernoulli distribution
   - **not true** for F^2 in general
2. cannot generate random words of even parity

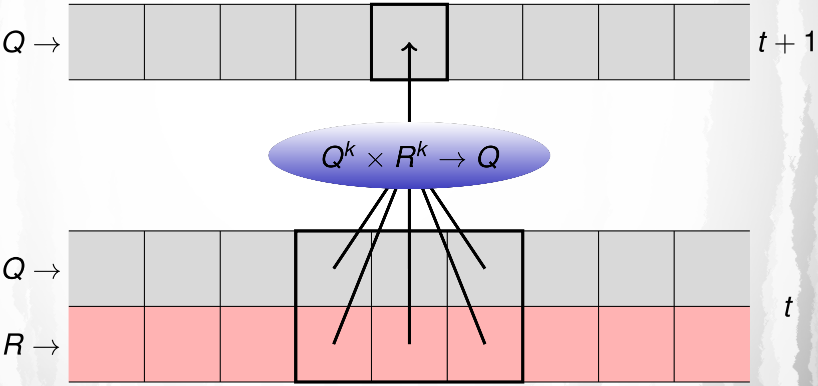


Fact

No correlation-free probabilistic CA can realize this function

Proposed formalism

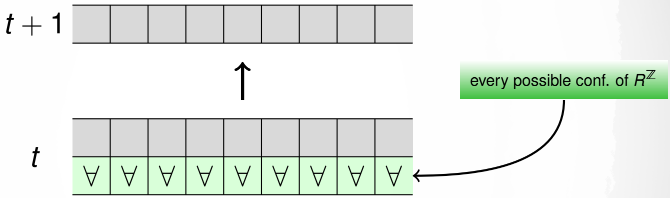
- (classical) states Q, k neighbors
- random states: R



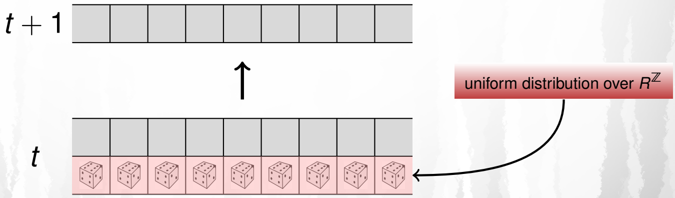
- explicit global map $F : Q^{\mathbb{Z}} \times R^{\mathbb{Z}} \rightarrow Q^{\mathbb{Z}}$

From local to global

1. **non-detreministic** global function $N_F : Q^{\mathbb{Z}} \to \mathcal{P}(Q^{\mathbb{Z}})$

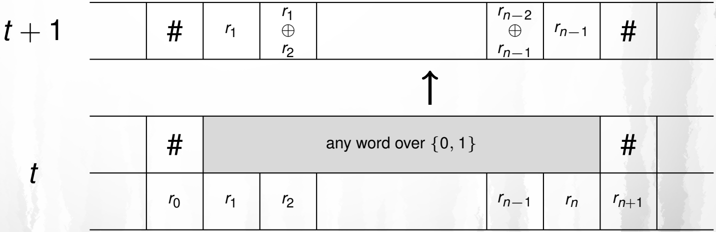


2. **stochastic** global function $S_F : Q^{\mathbb{Z}} \to \mathcal{M}(Q^{\mathbb{Z}})$



Usefulness of local correlations

- global iterations
  - by locality, we define (canonically) $S_F^t : Q^{\mathbb{Z}} \to \mathcal{M}(Q^{\mathbb{Z}})$
  - **Fact:** if $S_f = S_G$ then $S_F^t = S_G^t$
- generating random even words
  - $Q = \{\#, 0, 1\}$
  - $R = \{0, 1\}$
  - $k = 3$



Overview of the talk

1. Definitions
2. **Decision problems**
3. Simulations

From local to global

- **syntactic object:** local map $f : Q^k \times R^k \to Q^k$
- **semantic object:** global map $S_F : Q^{\mathbb{Z}} \to \mathcal{M}(Q^{\mathbb{Z}})$

Given a syntactic object,
can we decide properties about
the corresponding semantic object

- **example:** equality of global maps
  - *input:* f and g
  - *question:* do we have $S_F = S_G$?

Equality of global maps

A non-trivial problem

Theorem

In **dimension 2 and higher**, equality of stochastic global maps is undecidable

- reduction from surjectivity problem for deterministic CA
- surjective $\Leftrightarrow$ preserves uniform distribution
- also undecidabile for **non-deterministic** global maps

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Fact

For **correlation-free** probabilistic CA, this problem is **decidable** in any dimension

- point-wise coupling of global maps
  - $F : Q^{\mathbb{Z}} \times R_F^{\mathbb{Z}} \to \mathcal{M}(Q^{\mathbb{Z}})$
  - $G : Q^{\mathbb{Z}} \times R_G^{\mathbb{Z}} \to \mathcal{M}(Q^{\mathbb{Z}})$

Coupling

F and G are **coupled** on c by $\gamma \in \mathcal{M}(R_F^{\mathbb{Z}} \times R_G^{\mathbb{Z}})$ if:

1. $\pi_1(\gamma)$ is the uniform distribution over $R_F^{\mathbb{Z}}$
2. $\pi_2(\gamma)$ is the uniform distribution over $R_G^{\mathbb{Z}}$
3. $\gamma(\{(r_F, r_G) \in R_F^{\mathbb{Z}} \times R_G^{\mathbb{Z}} : F(c, r_F) = G(c, r_G)\}) = 1$

Equality of global maps

Equality vs. coupling

- point-wise coupling of global maps
  - $F : Q^{\mathbb{Z}} \times R_F^{\mathbb{Z}} \to \mathcal{M}(Q^{\mathbb{Z}})$
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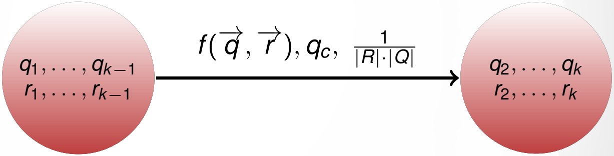
Theorem

$S_F = S_G$ if and only if F and G are coupled on each $c \in Q^{\mathbb{Z}}$.

Equality of global maps

1D, the realm of finite automata

- given $f : Q^k \times R^k \to Q$
- finite **probabilistic de Bruijn automaton** $\mathcal{A}_F$:
  - vertex set: $Q^{k-1} \times R^{k-1}$
  - edges labeled by $Q \times Q$, and weighted:



  - *NB: initialization part omitted here for simplicity*

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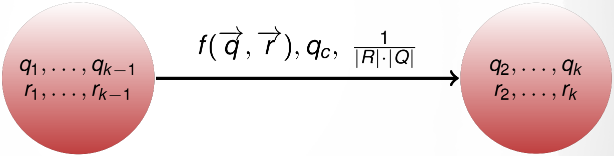

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Fact

$L(\mathcal{A}_F) = L(\mathcal{A}_G) \Leftrightarrow S_f = S_G$

Corollary

Equality of stochastic global maps is **decidable** in 1D.

Büchironnerie

Model checking for 1D Non-deterministic CA

- known proof technique for 1D deterministic CA

K. Sutner, 2009

Model Checking One-Dimensional Cellular Automata

O. Finkel, 2011

On Decidability Properties of One-Dimensional Cellular Automata

Key result

ω-automatic structures have a decidable first-order theory.

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Key result

ω-automatic structures have a decidable first-order theory.

- extension to non-detreministic 1D CA
  - reachability predicate: $x \to y \equiv y \in N_F(x)$
  - $(Q^{\mathbb{Z}}, \to)$ has a **decidable** first-order theory
- *injectivity:* $\forall x, y, z : (x \to z \wedge y \to z) \Rightarrow x = y$
- *determinism:* $\forall x, y, z : (x \to z \wedge x \to y) \Rightarrow y = z$
- *being noisy:* $\forall x, y : x \to y$

1D, still some undecidability

Simulating probabilistic automata by stochastic CA

- **given** a finite probabilistic automaton $\mathcal{A}$
  - alphabet Σ, states Q_A
  - $N =$ lcm of denominators of (rational) transition probas
- **construct** F with
  - $Q = \Sigma \cup \{0, a, b, c\}$
  - $|R| = N \cdot |Q_A|$
  - random $r \in R \leftrightarrow$ random pair (vertex,outgoing edge)

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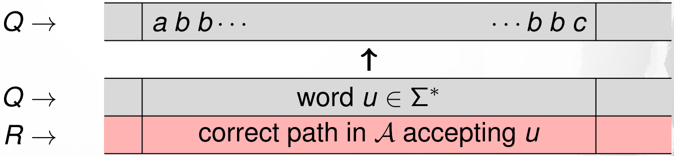

  - output 0 in any other case

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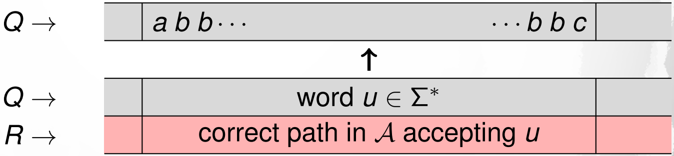

  - output 0 in any other case

Fact

For any $u \in \Sigma$, we have $Pr\{F(u) = ab^{|u|}c\} = Pr_{\mathcal{A}}(u) \cdot |Q_A|^{-|u|}$

1D, still some undecidability

Emptiness problem

- given $\mathcal{A}$ and $\lambda > 0$, is there *u* such that $Pr_{\mathcal{A}}(u) > \lambda$?

Blondel-Canterini, 2003

Undecidable Problems for Probabilistic Automata of Fixed Dimension

- undecidable problem, even for automata over $\{0, 1\}$ with 46 states

1D, still some undecidability

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Undecidable Problems for Probabilistic Automata of Fixed Dimension

- undecidable problem, even for automata over $\{0, 1\}$ with 46 states

Corollary

There is some α such that the following problem is undecidable in 1D: given a stochastic CA *F* and $\lambda > 0$, is there some word *u* such that

$$Pr\{F(u) = ab^{|u|}c\} > \lambda \cdot \alpha^{|u|}$$

- the non-probabilistic version is decidable
- conjectured also decidable for correlation-free CA

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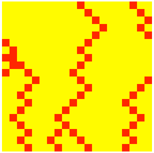
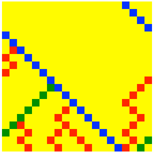
Restrictions/projections

- **restriction** to a stable subset of states: $F \sqsubseteq G$



$\sqsubseteq$



$i : Q_F \to Q_G$ with $i(G'(c,r)) = G(i(c), r)$ and $S_{G'} = S_F$

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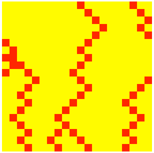

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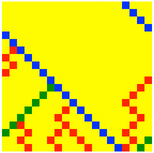
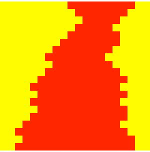
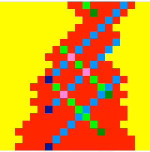


$i : Q_F \to Q_G$ with $i(G'(c,r)) = G(i(c),r)$ and $S_{G'} = S_F$

- compatible **projection**: $F \trianglelefteq G$



$\trianglelefteq$

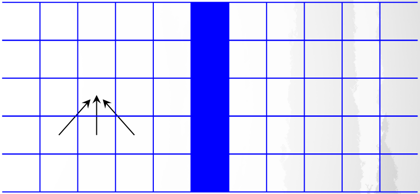


$\pi : Q_G \to Q_F$ with $G'(\pi(c),r) = \pi(G(c,r))$ and $S_{G'} = S_F$

- any combination of both: projection after restriction $\trianglelefteq\!\sqsubseteq$

Rescaling

- **3 parameters:** $F \mapsto F^{\langle m,t,z \rangle}$



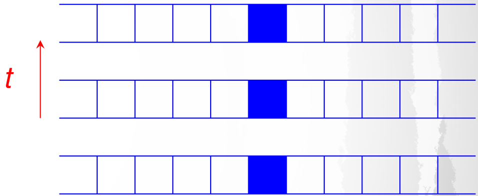
Global map

$$F^{\langle 1,1,0 \rangle} = F$$

Rescaling

▶ **3 parameters:** $F \mapsto F^{\langle m,t,z \rangle}$

■ time



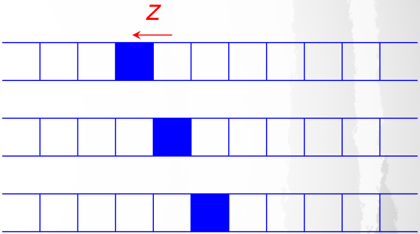
Global map

$$F^{\langle 1,t,0 \rangle} = F^{t}$$

Rescaling

▶ **3 parameters:** $F \mapsto F^{\langle m,t,z\rangle}$

- time
- shift

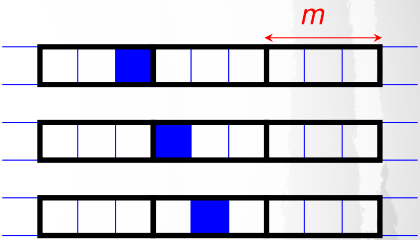


Global map

$$F^{\langle 1,t,z\rangle} = \sigma_z \circ F^t$$

Rescaling

- **3 parameters:** $F \mapsto F^{\langle m,t,z\rangle}$



- time
- shift
- cell grouping

Global map

$$F^{\langle m,t,z\rangle} = \mathbf{o}_m^{-1} \circ \sigma_z \circ F^t \circ \mathbf{o}_m$$

$$\left.\begin{array}{r} \mathbf{o}_m : (Q^m)^{\mathbb{Z}} \to Q^{\mathbb{Z}} \\ \mathbf{o}_m : (Q^m)^{\mathbb{Z}} \times (R^m)^{\mathbb{Z}} \to Q^{\mathbb{Z}} \times R^{\mathbb{Z}} \end{array}\right\} \text{canonical bijections}$$

Rescaling

▶ **3 parameters:** $F \mapsto F^{\langle m,t,z \rangle}$

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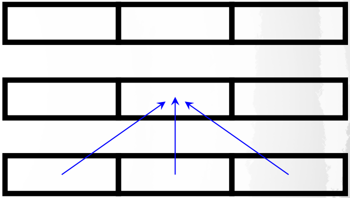

Global map

$F^{\langle m,t,z \rangle}$ is a cellular automaton

- with possibly different state sets (Q and R)
- with a possibly different neighborhood

Simulation and universality

- simulation $\equiv$ local comparison **up to** rescaling
- $F \preceq_i G$ **iff** $\exists m, m', t, t', z, z' : F^{\langle m,t,z \rangle} \sqsubseteq G^{\langle m',t',z' \rangle}$
- $F \preceq_\pi G$ **iff** $\exists m, m', t, t', z, z' : F^{\langle m,t,z \rangle} \trianglelefteq G^{\langle m',t',z' \rangle}$
- analogously, define $\preceq_i^N$ and $\preceq_\pi^N$ for non-deterministic maps

  (asking $N_{G'} = N_G$ instead of $S_{G'} = S_G$ in def. of $\sqsubseteq$ and $\trianglelefteq$)
- **Fact:** if $F \preceq G$ then $F \preceq^N G$ for $\preceq = \preceq_i$ or $\preceq_\pi$

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- **Fact:** if $F \preceq G$ then $F \preceq^N G$ for $\preceq = \preceq_i$ or $\preceq_\pi$

Universality

Let $\preceq$ be any of $\preceq_i$, $\preceq_\pi$, $\preceq_i^N$ or $\preceq_\pi^N$.
G is $\preceq$-universal if for all F it holds $F \preceq G$

Universality

- **definition:** $\mathcal{PF}(F)$ is the set of prime factors of $|R_F|$

Proposition

If F and G are not deterministic and $F \preceq_i G$ (or $F \preceq_\pi G$) then $\mathcal{PF}(F) \cap \mathcal{PF}(G) \neq \emptyset$

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Corollary

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Theorem

1. There is a $\preceq_i^N$-universal CA.
2. for any set of prime number P, there is a **correlation-free** CA G such that for any F

$$\mathcal{PF}(F) \subseteq P \Rightarrow F \preceq_i G$$

Correlation-free vs. general CA

Proposition

For any F there is a **correlation-free** G such that $F \sqsubseteq G^2$

- **undecidable in 2D:** F, G correlation-free, do we have $S_F^2 = S_G^2$?
- can we do anything with 2 steps of a correlation-free CA?

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- can we do anything with 2 steps of a correlation-free CA?

NO!

There is no correlation-free F and t such that F^t is the 'random even word function'.

- the set of space-time diagrams ($\subseteq Q^{\mathbb{N}\times\mathbb{Z}}$) of a correlation-free CA is a SFT
- the set of space-time diagrams of a CA is a sofic shift

Questions

Questions

- is there for any F a correlation-free G such that

$$F \trianglelefteq G$$

- $\preceq_i$ equivalence classes without correlation-free CA?
- more on decidability/undecidability in stochastic 1D maps
- continuity/computability of coupling
- find a formalism of PCA with a Hedlund theorem

Merci !