Probabilistic Cellular Automata with Erosion

Exponential Decay of Correlations in the Low-Noise Regime

Lise Ponselet

Université Catholique de Louvain, Belgium
FNRS aspirant

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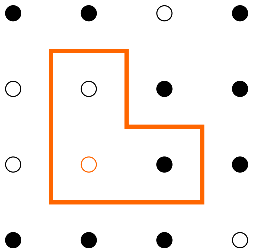
Joint work with Augustin de Maere

Outline

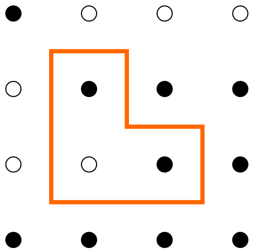
1. Cellular automata with erosion
   - The Toom model
   - The Stavskaya model
   - An erosion criterion
2. Low-noise regime
   - Stability of homogeneous configurations
   - Exponential convergence to equilibrium

The North-East-Center majority rule



- $\mathbb{Z}^2$ lattice
- Synchronous updates
- CA updating rule: majority among three neighbours
- PCA: at each time step and at each site, colour flip with probability ϵ in $[0, 1]$

The North-East-Center majority rule



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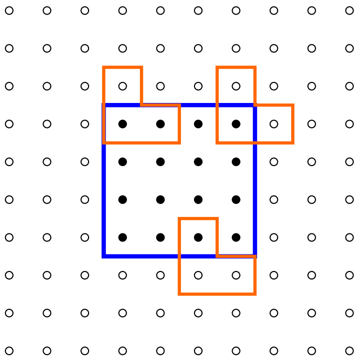
Erosion of finite • blocks

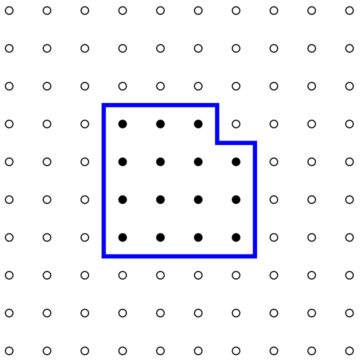


For $\epsilon = 0$, deterministic evolution...

... any finite block of ●
surrounded with a sea of ○
is erased in a finite time.

Question: when $\epsilon > 0$, does the ○ phase survive thanks to this erosion property?

Erosion of finite • blocks



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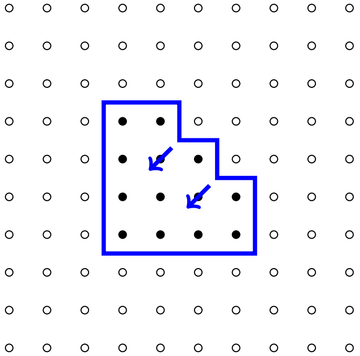

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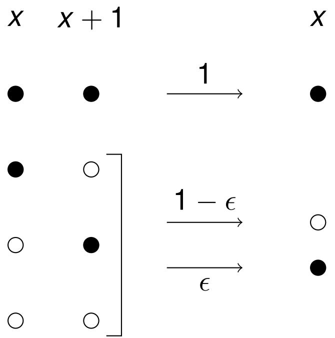
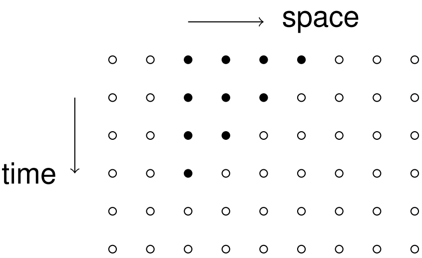
A toy model: the Stavskaya rule



- $\mathbb{Z}$ lattice
- Synchronous updates
- CA updating rule:
  one-sided contact process
- PCA: errors create only ●.

Erosion of finite • blocks



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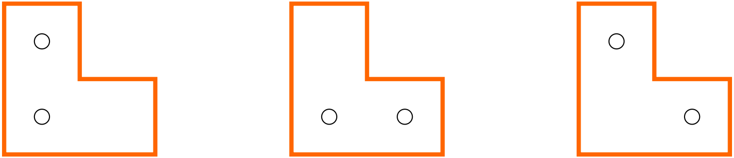
An erosion criterion

Minimal ○ subsets of the neighbours' set:



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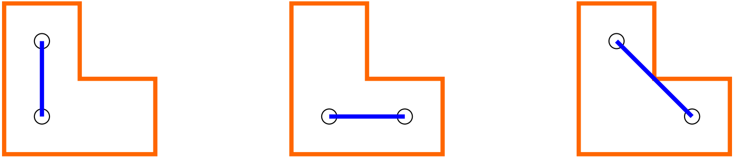
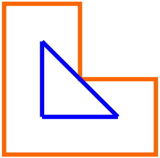


The convex hulls
of all minimal ○ subsets
have an empty intersection.



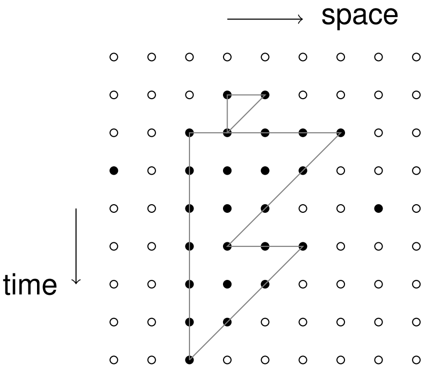
Stability of the ○ configuration

Theorem (Toom, 1978)

For the Stavskaya model, the North-East-Center Toom model and a whole class of PCA, namely the monotonic binary tessellations with the erosion property, if the initial condition has ○ everywhere, then the probability of finding a ● at any given site of the space-time lattice tends uniformly to 0 *as* ϵ *tends to* 0.

Consequence:
phase transition for the Stavskaya model and the NEC model

Proof method: contour expansion

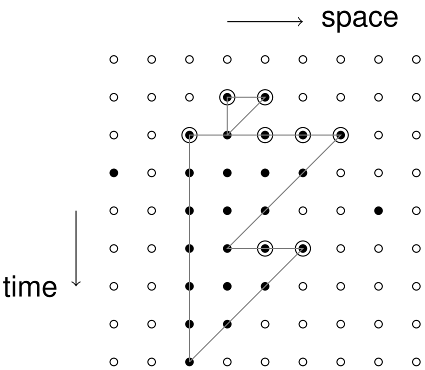


Competition between

- large number of possible contours: entropic factor
- low probability of making all associated errors: energetic factor

The energetic factor wins if ϵ is small enough.

Proof method: contour expansion



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Proof method: graph expansion

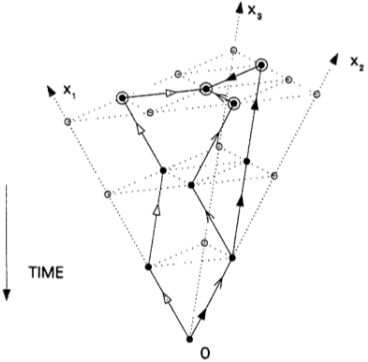


Figure: Lebowitz-Maes-Speer, 1990

Competition between

- large number of possible graphs: entropic factor
- low probability of making all associated errors: energetic factor

The energetic factor wins if ϵ is small enough.

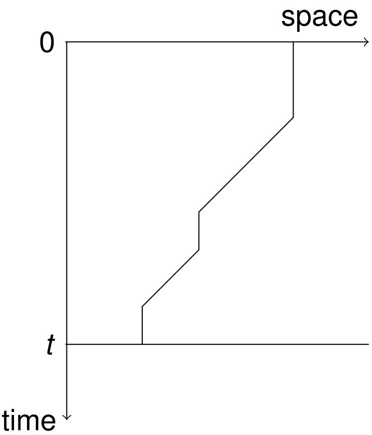
Exponential convergence to equilibrium

Theorem

For the Stavskaya model, the North-East-Center Toom model and a whole class of PCA, namely the monotonic binary tessellations with the erosion property, if the initial condition has $\circ$ *everywhere, then for* ϵ *small enough,*

- *the system converges exponentially fast to equilibrium;*
- *the equilibrium state has exponential decay of correlations in space and in time.*

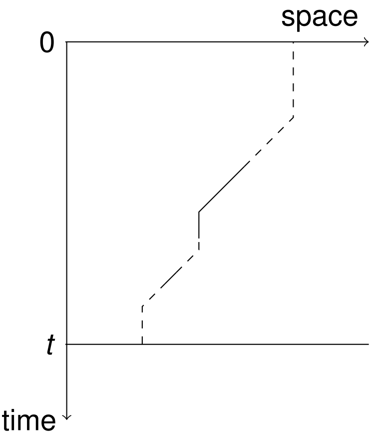
Proof method: paths and graphs



Expansion in terms of paths and graphs: sum over

- space-time paths of influence between neighbours
- partitions of the paths into segments crossing the ○ phase and segments crossing the ● phase
- graphs attached to the paths in the ● phase

Proof method: paths and graphs



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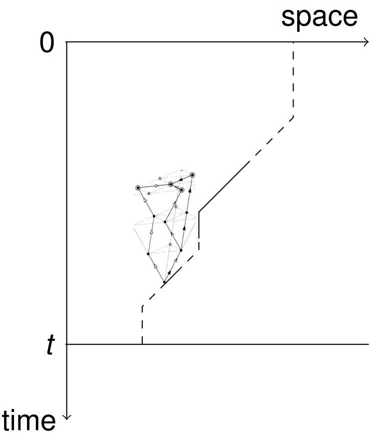

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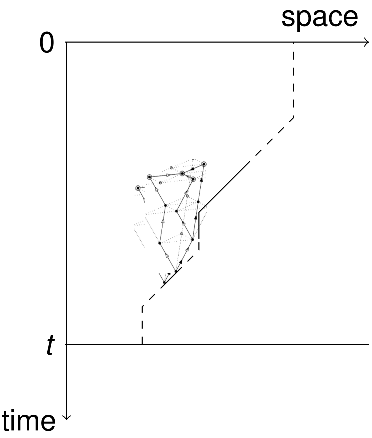

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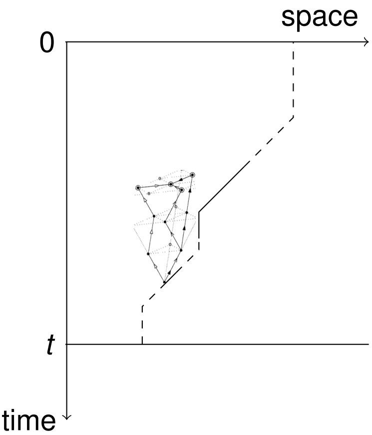

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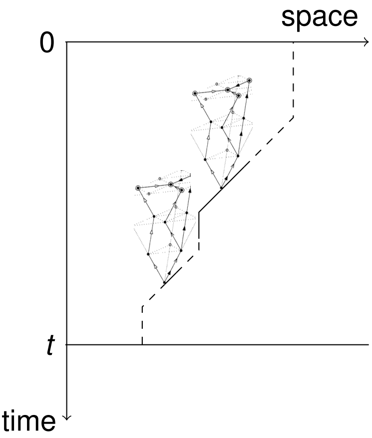

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Summary

- Some updating rules for CA lead to deterministic erosion of finite blocks of one colour in a sea of the opposite colour.
- The space-time configuration with a highly predominant colour is then stable under the introduction of a small error rate.
- Graph expansions provide a fruitful tool to explore the properties of extremal phases in the low-noise regime.

- Outlook
  - PCA with more than two colours
  - PCA without erosion

For Further Reading

A. Toom.
Cellular automata with errors: problems for students of probability.
In L. Snell (ed.) Topics in Contemporary Probability and its Applications. Probability and Stochastics Series.
CRC Press, Boca Raton, 1995.

A. de Maere and L. Ponselet.
Exponential decay of correlations for strongly coupled Toom probabilistic cellular automata.
J. Stat. Phys., 147(3):634–652, 2012.