Probabilistic Cellular Automata: a statistical mechanics point of view

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Outline

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2. Static Probability models for interacting sites: Gibbs measures
3. Loss of ergodicity at infinite volume for a potential-related family of PCA dynamics
   - Motivation of the class of reversible PCA dynamics
   - Finitely many interacting sites (finite volume)
   - Infinitely-many interacting sites (infinite volume)
4. Synchronisation for PCA with spin space $[0, 1]$

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2. Static Probability models for interacting sites: Gibbs measures
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Probabilistic Cellular Automata

Terminology: PCA = random CA = stochastic CA
= locally interacting Markov chains

- $G = (V, E)$ is a graph whose nodes are the elementary entities
  usually $G = \mathbb{Z}^d$ or $G = \Lambda \Subset \mathbb{Z}^d$
- S a finite space (possible states of an entity)
  e.g. $S = \{0, 1\}$, $S = \{-1, +1\}$...
- $\eta := (\eta_k)_{k \in \Lambda}$ is called a configuration
- local updating stochastic rule: $p_k(s|\eta)$, $k \in G$, $s \in S$, $\eta \in S^V$
    - $p_k(\cdot\,|\sigma)$ is a probability on S
    - $\exists\ V_k \Subset V$

$$p_k(s|\eta) = p_k(s|\eta_{V_k})$$

Mathematical Probability framework

- it is a stochastic Markov process $(\sigma(n))_{n\in\mathbb{N}}$ with discrete-time on the state space S^Λ
- challenge: understand the dynamical evolution in a statistical point of view
- A state of the system is a probability measure on the configurations space S^V.
- P is said to be a (finite volume) PCA (synchronous) dynamics if

$$P^\tau_\Lambda(\eta,\sigma) = \prod_{k\in\Lambda} p_k(\sigma_k|\eta_\Lambda\tau_{\Lambda^c})$$

- P is said to be a (infinite volume) PCA dynamics if

$$P(\eta,\sigma) = \bigotimes_{k\in\mathbb{Z}^d} p_k(\sigma_k|\eta)$$

Some historial references following this point of view

- Piatetski-Shapiro
- Toom, Vasilyev, Stavskaja, Mitjushin, Kurdumov, Pirogov, 1970–1978; Dawson, 1974
- Kozlov, Vasilyev, 1980
- Künsch, 1984; Malyshev,Ignatyuk, 1987
- Georges,Le Doussal, 1989
- Lebowitz, Maes, Speer, 1990 – Maes, Shlosman 1991

Some remarks

- no general theory, no general results
- only few general results from Markov Processes general theory
- statistical point of view on the dynamical evolution, long-time behaviour
- state of the system is a probability distribution μ on S^Λ
- candidates for equilibrium/steady states reached in the long-time behaviour are stationary distributions μ which are probability distributions invariant w.r.t. dynamics *i.e.*

$$\mathbb{P}(\sigma(0) = \cdot) = \mu(\cdot) \Rightarrow \forall n \geqslant 1,\ \mathbb{P}(\sigma(n) = \cdot) = \mu(\cdot)$$

or, equivalently $\mu P = \mu$

Detailed balance

- The Markov dynamics is said to be reversible if there exists a probability distribution μ such that

$$P(d\sigma|\eta)\mu(d\eta) = P(d\eta|\sigma)\mu(d\sigma).$$

It means, that the distribution of the process, starting with μ as initial distribution is invariant under time-reversal.

- In particular, such a distribution is a stationary distribution.

Example: Stavskaja Model on $\mathbb{Z}$

$S = \{0,1\}$, $G = \mathbb{Z}$, $V_k = \{k-1, k\}$

$$p_k(1|\sigma) = \begin{cases} 1 \text{ if } \sigma_k = \sigma_{k-1} = 1, \\ \varepsilon \in [0,1] \text{ otherwise.} \end{cases}$$

- finitely-many interacting sites $\Lambda \Subset \mathbb{Z}$: since $\sigma \equiv 1$ is absorbing, long-time behaviour is absorption in $\delta_{\underline{+1}}$
- infinitely-many interacting sites $\Lambda = \mathbb{Z}$, it exists $\varepsilon^* > 0$ such that
  - with $\varepsilon > \varepsilon^*$, the dynamics is ergodic

    $$\forall \mu \text{ starting distribution, } \lim_{n\to\infty} \mathbb{P}_\mu(\sigma(n) = \cdot) = \delta_{\underline{+1}}$$

  - with $\varepsilon < \varepsilon^*$, the dynamics is non-ergodic, in particular

    $$\lim_{n\to\infty} \mathbb{P}_{\delta_{\underline{0}}}(\sigma(n) = \cdot) = \mu_\varepsilon(\cdot) \neq \delta_{\underline{+1}}.$$

    and every stationary distribution translation-invariant is of the form

    $$\alpha\mu_\varepsilon + (1-\alpha)\delta_{\underline{+1}}, \quad \alpha \in [0,1].$$

See [Leontovitch & Vaserstein, 1970] and more new results [Mendonça 2011]

Questions

- May be the non-trivial μ_ε is due to the deterministic component of the updating rule ?
- How are these stationary distributions like ?
- Why do we care about infinitely-many interacting entities ?

Static probability models for interacting sites

Aim: try to find candidates and characterise the equilibrium distributions of these dynamics, on the infinite space $S^{\mathbb{Z}^d}$

$E = S^\Lambda$, $\Lambda \Subset \mathbb{Z}^d$, $S = \{-1, +1\}$, $|E| = 2^{|\Lambda|}$

Energy $H_\Lambda(\eta) = -\sum_{i \sim j} \; J_{i,j} \; \eta_i \eta_j$

$$\mathbb{P}^\tau_\Lambda(\{\omega : \sigma(\omega) = (\eta_k)_{k\in\Lambda}\}) = \frac{1}{\mathcal{Z}^\tau_\Lambda} e^{-\beta H_\Lambda(\eta_\Lambda \tau_{\Lambda^c})}, \qquad \mathcal{Z}^\tau_\Lambda = \sum_{\eta_\Lambda \in E} e^{-\beta H_\Lambda(\eta_\Lambda \tau_{\Lambda^c})}$$

with the inverse temperature parameter $\beta > 0$.
Ising potential $\varphi_{i,j}(\sigma) = J_{i,j}\sigma_i\sigma_j$ when $i \sim j$.

Static probability models for interacting sites: Gibbs distributions

- finitely-many interacting sites

$$\mu_\Lambda^\tau(\sigma_\Lambda) = \frac{1}{\mathcal{Z}_\Lambda^\tau} e^{-\beta H_\Lambda(\sigma_\Lambda \tau_{\Lambda^c})}$$

Remark: Compatibility property : $\Lambda_2 \subset \Lambda_1$

$$\mu_{\Lambda_1}^{\tau_{\Lambda 1^c}}(\sigma_{\Lambda_2} | \sigma_{\Lambda_1 \setminus \Lambda_2}) = \mu_{\Lambda_2}^{\tau_{\Lambda 1^c} \sigma_{\Lambda_1 \setminus \Lambda_2}}(\sigma_{\Lambda_2})$$

- infinitely-many interacting sites, $\forall \Lambda \Subset \mathbb{Z}^d$, $\forall \tau$ boundary condition,

$$\mu(\ (\cdot)_\Lambda \mid \tau_{\Lambda^c}\ ) = \mu_\Lambda^\tau(\ \cdot\ )$$

Main motivation: construct distributions on $S^{\mathbb{Z}^d}$, which are “locally” the same.

When the number of interacting sites goes to infinity (thermodynamic limit), there may be different limiting distributions: phase transition.

Relation IPS/Gibbs measure

Let μ be a Gibbs measure on $S^{\mathbb{Z}^d}$ w.r.t. a potential $(\varphi_A)_{A\subset\mathbb{Z}^d}$.
it exists a continuous-time Markov process on $S^{\mathbb{Z}^d}$ with a sequential updating rule (*interacting particle system dynamics*), such that

$$\mathcal{R} = \mathcal{G}(\varphi), \qquad \mathcal{S}_i \subset \mathcal{G}(\varphi).$$

Ergodicity of the dynamics $\rightsquigarrow$ Phase transition w.r.t. $(\varphi_A)_{A\subset\mathbb{Z}^d}$.

Specificities of the synchronous updating

- the existence of some reversible steady state costs more than for a sequential updating dynamics: the reversed probability kernel is required to be of product form
- for some given Gibbs measure μ, on the contrary to Markov dynamics with sequential updating, there is no canonical way to construct a PCA dynamics, which would admit μ as stationary measure.

Proposition (*Dawson, 1975*)

Let μ be a Gibbs measure with respect to some nearest neighbour potential on $\{0,1\}^{\mathbb{Z}^2}$. There is no translation invariant PCA, which could have μ as reversible measure.

Associated space potential

Theorem (*P. Dai Pra / C. Maes, 1992*)

Let $P = (p_k)_{k \in \mathbb{Z}^d}$ be a PCA dynamics on $S^{\mathbb{Z}^d}$, translation invariant and purely stochastics

$$p_k(s|\sigma) > 0$$

if $\exists \varphi$ potential on $S^{\mathbb{Z}^d}$ s.t. $\mathcal{G}(\varphi) \cap \mathcal{S}_i \neq \varnothing$ then $\mathcal{S}_i \subset \mathcal{G}(\varphi)$.

Outline

Motivation for another PCA family of dynamical models

- There are techniques to write down Markov processes associated to a given potential, such that the equilibrium distributions are characterised as Gibbs distributions w.r.t this potential:
  mainly Hastings-Metropolis algorithm (flip dynamics) and Gibbs-sampler.
- considering the case of one site $\Lambda = \{k\}$, the Gibbs-sampler for the Ising potential gives

$$\begin{aligned} \mu_{\{k\}}(\sigma_k|\sigma_{V_{k^c}}) &= \frac{\exp\Big(\big(\beta \sum_{j\sim k} \mathcal{K}(j-k)\sigma_k\sigma_j + \beta h \sigma_k)\big)\Big)}{2\cosh(\beta(\sum_{j\sim k} \mathcal{K}(j-k)\sigma_j + h))} \\ &= \frac{1}{2}\Big(1 + \sigma_k \ \tanh(\beta(\sum_{j\sim k} \mathcal{K}(j-k)\sigma_j + h))\Big) \end{aligned}$$

A family of reversible PCA dynamics on $\{-1,+1\}^{\mathbb{Z}^d}$

Define, for $k \in \mathbb{Z}^d$, $s \in S$, $\eta \in S^Z d$

$$p_k(\ s\ |\eta) = \frac{1}{2}\Big(1 + \ s\ \tanh\big(\beta(\sum_j \mathcal{K}(j-k)\eta_j + h)\big)\Big)$$

where

- $\beta \geqslant 0$ (tunning the spatial and temporal dependence)
- $\mathcal{K} : \mathbb{Z}^d \to \mathbb{R}$ is symmetric and with finite range

$$V_k := \{j \in \mathbb{Z}^d, \mathcal{K}(j-k) \neq 0\} \Subset \mathbb{Z}^d$$

- $h \in \mathbb{R}$

Note, it is a purely stochastic dynamics.

$$\forall k, \forall s \in S, \forall \eta \in S^{\mathbb{Z}^d}, \quad p_k(s|\eta) > 0$$

It is the form of a general reversible PCA dynamics on $\{-1,+1\}^{\mathbb{Z}^d}$.

Outline

Identification of the finite-volume steady states

For the PCA P_Λ^τ on S^Λ with fixed boundary condition τ there exists a unique stationary (reversible) prob. measure ν_Λ^τ given by

$$\nu_\Lambda^\tau(\eta_\Lambda) = \frac{1}{\mathcal{W}_\Lambda^\tau} \prod_{k\in\Lambda} e^{\beta h \eta_k} \cosh\left[\beta \sum_{j\in\mathbb{Z}^d} \mathcal{K}(j-k)(\eta_\Lambda \tau_{\Lambda^c})_j + \beta h\right] e^{\beta \eta_k \sum_{j\in\Lambda^c} \mathcal{K}(k-j)\tau_j}$$

and with periodic boundary condition:

$$\nu_\Lambda^{\mathrm{per}}(\eta_\Lambda) = \frac{1}{\mathcal{W}_\Lambda^{\mathrm{per}}} \prod_{k\in\Lambda} e^{\beta h \eta_k} \cosh(\beta \sum_{j\in\mathbb{Z}^d} \mathcal{K}(j-k)(\eta_\Lambda^{\mathrm{per}})_{k'})$$

$$\text{rewrite } \nu_\Lambda^{\mathrm{per}}(\eta_\Lambda) = \frac{1}{\mathcal{W}_\Lambda^{\mathrm{per}}} e^{-\sum_{V_k\cap\Lambda\neq\varnothing} \varphi_{V_k}(\eta^{\mathrm{per}})} \qquad \text{with}$$

$$\varphi_{V_k}(\eta) = -\log\cosh(\beta \sum_{j\in V_k} \mathcal{K}(j-k)\eta_j + \beta h) \qquad \varphi_{\{k\}}(\eta) = -\beta h \eta_k$$

$\nu_\Lambda^{\mathrm{per}}$ is the finite volume Gibbs measure with periodic boundary associated with the potential φ.

Effect of the synchronous updating scheme on the stationary distribution

As consequence: when the update probability $p_k(s|\eta)$ is applied

- with sequential updating scheme: it is the Hastings-Metropolis Markov chain (coincide with the Gibbs sampler), the dynamics converges toward the Gibbs measure associated with the n.n. Ising potential
- with parallel updating scheme: PCA case, the dynamics converges (for periodic boundary conditions) towards the Gibbs measure related to the potential φ

$$\varphi_{V_k}(\eta) = -\log \cosh(\beta \sum_{j \in V_k} \eta_j)$$

Fact:

For some given probability updating rule, changing the updating scheme modifies the nature of the stationary measure.

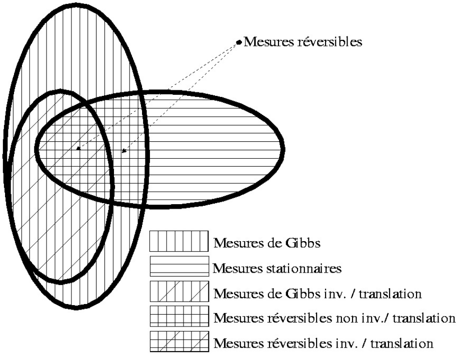
Outline

Characterisation of stationary meas. as Gibbs measures

(*Kozlov-Vasilyev, 1980; Künsch 1984; Dai Pra 1992; Dai Pra, Louis, Rœlly 2002*)

Relations between $\mathcal{R}, \mathcal{S}$ and $\mathcal{G}(\varphi)$:

$$\mathcal{R} = \mathcal{G}(\varphi^P) \cap \mathcal{S}, \qquad \mathcal{R}_i = \mathcal{S}_i.$$



Ergodicity criterion for P.C.A.

Dobrushin-Vasershtein Criterion

If

$$\gamma := \sup_{k\in\mathbb{Z}^d} \sum_{j\in V_k} \sup_{\sigma_{V_k}\in S^{V_k}} \left| p_k(+1|\sigma^j_{V_k}) - p_k(+1|\sigma_{V_k}) \right| \qquad < 1$$

then the PCA on $S^{\mathbb{Z}^d}$ is ergodic, more precisely
$\forall f \ : \ S^{\mathbb{Z}^d} \to \mathbb{R}, \ \mathrm{local} \ , \exists \ c(f) \in \mathbb{R}, \ \forall n \in \mathbb{N}^*,$

$$\sup_{\sigma\in S^{\mathbb{Z}^d}} \left| \mathbb{E}(\ f(\omega(n)) \mid \omega(0) = \sigma \ ) - \mathbb{E}_\nu(f) \right| \leqslant \frac{c(f)}{1-\gamma}\gamma^n$$

In the particular case presented $\gamma = 2\tanh(2\beta)$, which means ergodicity for $\beta < \frac{1}{2}\mathrm{Argtanh}(\frac{1}{2}) \simeq 0,27$

Theorem [Dai Pra, L., Rœlly 2002 and L. 2004 & 2005]

Characterisation of the dynamics' ergodicity through the potential

Let P be a PCA on $\{-1,+1\}^{\mathbb{Z}^d}$ from the previous family s.t. $\mathcal{K}(.) \geqslant 0$.

- dynamics P is ergodic $\iff$ there is no phase transition (w.r.t φ)
- moreover $\exists C > 0,\ \exists M > 0,\ \exists L_1 \in \mathbb{N}^*, \forall L \in \mathbb{N}^*, L \geqslant L_1$,

  $$0 \leqslant \mathbb{E}_{\mu((\cdot)_{\mathcal{B}_L}|\sigma_{\mathcal{B}_L{}^c}=+1)}(\sigma_0) - \mathbb{E}_{\mu((\cdot)_{\mathcal{B}_L}|\sigma_{\mathcal{B}_L{}^c}=-1)}(\sigma_0) \leqslant Ce^{-ML}$$

  is equivalent to $\exists \lambda > 0,\ \forall n \geqslant n_1,\ \forall\ f$ local function on $S^{\mathbb{Z}^d}$,

  $$\sup_\sigma \left|\mathbb{E}_{\delta_\sigma}(f(\sigma(n))) - \mathbb{E}_\mu(f)\right| \leqslant 2\ |||f|||e^{-\lambda n}$$

- $d = 2$, $\exists \beta_0,\ \forall \beta \geqslant \beta_0$ it exists different stationary distribution, which are also (infinite volume) Gibbs measures (w.r.t φ) (*i.e.* there is phase transition)

Main points, answers to the preliminary questions

- class of synchronous stochastic dynamics related to a potential, characterisation of stationary distribution as Gibbs distributions
- interplay between static and dynamical results
  - ergodicity means absence of sensibility of the starting distribution
  - phase transition related to sensibility to spatial boundary conditions

Main points, answers to the preliminary questions

- going from sequential to parallel updating scheme change the associated potential
- different time-asymptotic behaviour in law between finite-volume and infinite,
  like in the Stavskaja PCA case, even for purely stochastic dynamics,
  BUT: have to go in dimension 2

Mean-field interacting Polya urns

- single Polya urn
- interacting Polya urns through mean-field
  finitely many urns $1 \leqslant i \leqslant N$,
  starting with $X_0(i) = 1$, discrete time $t \in \mathbb{N}$,

$$X_{t+1}(i) = X_t(i) + Y_t(i)$$

  where,
  - $X_t(i)$ is the number of black balls in urn i, at time t
  - $Z_t(i) := \frac{X_t(i)}{t+2}$ is the fraction of black balls
  - $Z_t := \frac{1}{N} \sum_{i=1}^{N} Z_t(i)$ is the global fraction
  - given $\mathcal{F}_t$, $Y_t(i)$ Bernoulli is distributed with parameter $\alpha Z_t + (1-\alpha) Z_t(i)$ where $\alpha \in [0,1]$
  - given $\mathcal{F}_t$, $Y_t(i)$ independent from $Y_t(j)$ with $j \neq i$

Almost sure synchronization [Dai Pra, L., Minelli 2013]

- (Z_t) is a bounded Martingale, thus it converges towards Z_∞
- as soon as $\alpha > 0$, for each $i = 1, 2, \ldots, N$,

$$\lim_{t \to +\infty} Z_t(i) = Z_\infty \qquad \text{almost surely}$$

- since $\mathbb{E}(Z_\infty^2) < \mathbb{E}(Z_\infty) = \frac{1}{2}$, there is no fixation