Cellular automata and percolation: an overview of selected connections

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Abstract. Cellular automata and percolation theory have been mutually enriching for some time now. Percolation theory studies the connectivity properties of random subgraphs of a regular network, and provides tools for studying probabilistic cellular automata. In some cases, it is also a great help for analysing the evolution of deterministic cellular automata which evolve from random configurations. Conversely, the theory of cellular automata provides new insights into some percolation properties, and raises many questions. In the present survey article, we recall some well-known connections between cellular automata and percolation, together with a more recent development, the study of a percolation game.

Keywords: cellular automata · percolation.

In the last decades, various works have highlighted some connections between cellular automata and percolation [12,20,21,22]. The aim of this article is to give a few examples illustrating the different forms these connections can take. To begin with, let us introduce cellular automata and percolation in a few words.

A cellular automaton (CA) is a dynamical system defined on the set $S^{\mathbb{Z}^d}$, for some finite set S of *symbols* (or *states*). It is obtained by iterating a local update rule simultaneously at every site of the lattice. Formally, for $n_1, \dots, n_m \in \mathbb{Z}^d$, the *cellular automaton* (CA) of *neighbourhood* $\mathcal{N} = \{n_1, \dots, n_m\}$ and local rule $f : S^m \to S$ is the function: $F : S^{\mathbb{Z}^d} \to S^{\mathbb{Z}^d}$ defined by:

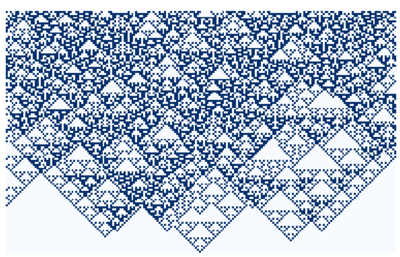
$$\forall x \in S^{\mathbb{Z}^d},\ \forall k \in \mathbb{Z}^d,\ F(x)_k = f(x_{k+n_1}, \dots, x_{k+n_m}).$$

For probabilistic cellular automata (PCA), all the cells are still updated simultaneously in discrete time, but the local rule is now a function $\varphi : S^m \to \mathcal{M}(S)$, where $\mathcal{M}(S)$ denotes the set of probability distributions on S. From a configuration $x \in S^{\mathbb{Z}^d}$, cell $k \in \mathbb{Z}^d$ is updated by a symbol chosen according to the distribution $\varphi(x_{k+n_1}, \dots, x_{k+n_m})$, independently for different cells. The local function φ allows one to define a global map $\Phi : \mathcal{M}(S^{\mathbb{Z}^d}) \to \mathcal{M}(S^{\mathbb{Z}^d})$, describing the action of the PCA on a configuration drawn according to a certain probability distribution.

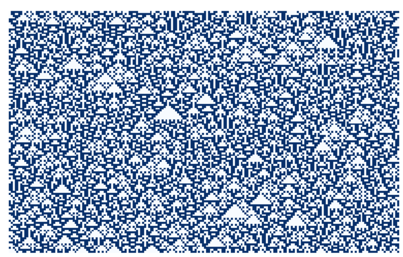
As an example, let us fix a value $\varepsilon \in [0,1]$, and consider the one-dimensional binary CA of neighbourhood $\mathcal{N} = \{-1, 0, 1\}$, defined by the local rule:

$$\varphi(x,y,z) = (1-\varepsilon)\,\delta_{x+z} + \varepsilon\,\delta_{x+z+1} = \begin{cases} x+z \text{ with probability } 1-\varepsilon \\ x+z+1 \text{ with probability } \varepsilon \end{cases},$$

where the sums $x+z$ and $x+z+1$ are computed modulo 2. The local rule thus consists in computing the sum (modulo 2) of the left neighbour and of the right neighbour, but with a probability ε of making a mistake, independently for different cells. For $\varepsilon = 0$, we recover a deterministic CA (rule 90 in Wolfram's notation), whose space-time diagram reveals a Sierpiński triangle when the initial configuration contains a single 1. Figure 1 shows two examples of space-time diagrams for the parameter $\varepsilon = 0.001$, from different initial configurations. Observe that if we erase the first few lines of the space-time diagram, it is quite difficult to have an idea of what the initial configuration was. The *noisy sum PCA* can indeed be shown to be ergodic for any $\varepsilon \in (0,1)$, see Section 2 for a formal definition. Moreover, its unique invariant measure is the uniform distribution [22].



Initial config. with a single 1



Initial config. distributed unif. at random

Fig. 1. Two space-time diagrams of the noisy sum PCA of parameter $\varepsilon = 0.001$, from different initial configurations (time goes up).

Percolation theory studies the properties of connected clusters in a random subgraph of a lattice. Let us label each cell of the grid $\mathbb{Z}^d$ independently by a 1 with probability p, and by a 0 with probability $1-p$, for some parameter $p \in (0,1)$. For p sufficiently large, the random configurations obtained contain almost surely an infinite path of cells labeled by 1, while for p small, the islands of cells in state 1 are almost surely all finite, see Fig. 2 for an illustration with $d = 2$. Among other questions, percolation theory is interested in investigating the critical value of the parameter p above which there exists almost surely such an infinite cluster.

Given a neighbourhood $\mathcal{N} = \{n_1, \ldots, n_m\}$, one can also consider the directed graph with set of vertices $\mathbb{Z}^d \times \mathbb{N}$, and bonds from (k,t) to $(k-n_i, t+1)$, for $k \in \mathbb{Z}^d, t \in \mathbb{N}$ and $n_i \in \mathcal{N}$. Again, we can label each vertex of $\mathbb{Z}^d \times \mathbb{N}$ independently by a 1 with probability p, and ask whether there exist infinite directed paths of vertices in state 1 in this directed percolation model, see Fig. 3 for an illustration with $d = 1$ and $\mathcal{N} = \{-1, 0, 1\}$.

$p = 0.4$ $p = 0.5$ $p = 0.6$

Fig. 2. Undirected site percolation on $\mathbb{Z}^2$: connected component of the origin for random configurations with different parameters.

$p = 0.4$ $p = 0.5$ $p = 0.6$

Fig. 3. Directed site percolation of neighbourhood $\mathcal{N} = \{-1, 0, 1\}$: connected component of the origin for random configurations with different parameters.

In Section 1, we study the evolution of some specific two-dimensional binary deterministic CA from random initial configurations. The first one is the *bootstrap CA*, which can be seen as a dynamical model of percolation. The second one is *Toom majority CA*, and exhibits a bifurcation behaviour, depending on whether the density of the initial configuration is greater or smaller than 1/2. The proof strongly relies on the fact that on the triangular lattice, the site percolation threshold is equal to 1/2.

In Section 2, we study the asymptotic behaviour of PCA, and show that properties of directed percolation on $\mathbb{Z}^d \times \mathbb{N}$ can be used to prove the ergodicity of a PCA on $\mathbb{Z}^d$.

Finally, in Section 3, we introduce and study a specific PCA that is related to the *hardcore model*, a model of lattice gas systems which has received much attention in statistical physics. We show that this PCA plays an unexpected role in the enumeration of *directed animals*, which are the possible shapes that directed percolation clusters can take, and that it is also involved in the study of a combinatorial game on percolation configurations.

We will sketch the strategies of the proofs without writing the proofs in detail. For further precision, the interested readers can consult the references that are provided all along the article. We shall generally refer to the book by Kůrka [16] and the survey by Kari [15] for background on deterministic cellular automata and to the surveys by Toom et al. [22] and Mairesse and Marcovici [18] for background on probabilistic cellular automata. The authoritative reference on percolation is the book by Grimmett [11].

1 Two well-known examples of CA related to percolation

In the following, we denote by $e_1 = (1, 0)$ and $e_2 = (0, 1)$ the two vectors of the standard basis of $\mathbb{R}^2$.

1.1 Bootstrap cellular automaton

Let us consider a binary CA whose local rule satisfies the following property: a cell in state 1 remains in state 1, and a cell in state 0 changes to a 1 if it has at least ℓ neighbours in state 1, for some fixed value ℓ. Such a CA can be seen as a simple model of infection spreading, where a cell becomes "infected" by contact with ℓ or more already infected neighbours, without possible remission. The two-dimensional case with $\ell = 2$ has attracted much attention. Precisely, let us define the *bootstrap CA* as the CA $F_B : \{0,1\}^{\mathbb{Z}^2} \to \{0,1\}^{\mathbb{Z}^2}$ defined on von Neumann neighbourhood $\mathcal{N} = \{0, \pm e_1, \pm e_2\}$ by the local rule

$$f_B((x_i)_{i\in\mathcal{N}}) = \begin{cases} 1 & \text{if} \quad x_0 = 1 \quad \text{or} \quad \mathrm{Card}\,\{i \in \mathcal{N} : x_i = 1\} \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Experimentally, when iterating this CA from an initial configuration where cells are in state 1 with probability p, independently for different cells, the state 1 invades the grid very quickly if p is large, while for p very small, we observe a stabilisation on a configuration made of disjoint rectangles of occupied cells. However, the following result was proved by van Enter [7].

Theorem 1. *For any $p > 0$, the bootstrap CA on the infinite lattice $\mathbb{Z}^2$ converges almost surely to the configuration "all 1", when iterated from an initial configuration where cells are in state 1 with probability p, independently for different cells.*

Sketch of the proof. The proof is based on the following observation: if there is somewhere in the configuration an occupied square (that is, a square whose cells are all in state 1) that is not surrounded by any empty rectangular contour (that is, a closed path of cells having a regular shape, whose cells are all in state 0), then we are sure that the state 1 will invade the grid. Even if p is very small, whatever the value of N, the probability of a square with side N being occupied is strictly positive, so that there exists almost surely such an occupied square in the configuration, as we work with infinite configurations. If N is sufficiently large, it is very likely that this occupied square has at least one cell in state 1 on its edge, and so on, resulting in the entire grid being invaded. $\square$

More recent work by Holroyd has led to a better understanding of what happens on finite grids [13]. In a nutshell, on a large $N \times N$ grid, if $p > \pi^2/(18 \log N)$, convergence to total occupancy occurs with high probability, while if $p < \pi^2/(18 \log N)$, it is not the case. However, one would need to do simulations on grids with a side of the order of 10^{20} to observe this threshold experimentally, as explained by Gravner and Holroyd [10].

1.2 Toom's majority CA

Another class of interesting binary CA consists in majority CA, for which the local rule is such that a cell changes to a 0 (resp. 1) if it has a strict majority of neighbours in state 0 (resp. 1). As a particular case, let us introduce Toom's majority CA: it is the binary CA $F_T : \{0,1\}^{\mathbb{Z}^2} \to \{0,1\}^{\mathbb{Z}^2}$ of neighbourhood $\mathcal{N} = \{0, e_1, e_2\}$ defined by the local rule

$$f_T(x_0, x_{e_1}, x_{e_2}) = \begin{cases} 1 & \text{if} \quad x_0 + x_{e_1} + x_{e_2} \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Unlike the bootstrap CA presented above, it presents a bifurcation phenomenon, according to the initial density of 1's [2].

Theorem 2. *Let us consider an initial configuration where cells are in state* 1 *with probability* p*, independently for different cells. If* $p < 1/2$*, Toom's majority CA converges almost surely to the configuration "all* 0*", while if* $p > 1/2$*, it converges almost surely to the configuration "all* 1*".*

Sketch of the proof. The proof of this result strongly relies on the fact that on the triangular lattice, the value of the site percolation threshold is exactly $1/2$. By symmetry, it is sufficient to prove the result for $p < 1/2$. Let us consider the (undirected) graph whose set of vertices is $\mathbb{Z}^2$, with the usual horizontal and vertical edges $\{x, x + e_1\}$ and $\{x, x + e_2\}$, for $x \in \mathbb{Z}^2$, together with NW-SE diagonal edges $\{x, x + e_2 - e_1\}$. We thus obtain a triangular lattice, and it is known that for $p < 1/2$, there exists almost surely no infinite 1-cluster in the initial configuration. Thus, the set of initially occupied cells consists almost surely of a countable union of finite 1-clusters. Toom's rule does not break up or connect different 1-clusters, and it is an *eroder*, which implies that any finite 1-cluster disappears in finite time. These ingredients allow one to demonstrate the desired result. $\square$

For a given $\varepsilon > 0$, let us consider the PCA that applies Toom's rule with probability $1-\varepsilon$, and do the contrary with probability ε, independently for different cells. For ε small enough, this PCA is known to have several invariant distributions. More precisely, it has at least one invariant distribution close to the configuration "all 0" and one close to "all 1". It thus provides a simple example of two-dimensional non-ergodic *positive-rate* PCA (all the probability transitions are strictly between 0 and 1). In dimension 1, it is much harder to exhibit positive-rate PCA that are non-ergodic: the only known example is due to Gács [9], and is very complicated. Also, it is on open problem to know whether there exists a one-dimensional CA that presents the same bifurcation phenomenon as Toom's CA. This is related to the density classification problem [2].

2 Ergodicity of PCA and directed percolation

A PCA is said to be *ergodic* if it asymptotically "forgets" its initial condition, meaning that the trajectories always converge to the same distribution regardless of the initial configuration. Formally, a PCA Φ is ergodic if it has a unique invariant probability distribution π that attracts every initial distribution, in the sense that for any distribution μ of $\mathcal{M}(S^{\mathbb{Z}^d})$, $\mu\Phi^t \to \pi$ weakly as $t \to +\infty$.

We will see that a comparison with directed percolation of same neighbourhood provides a sufficient condition for the ergodicity of PCA. First, let us come back to the directed percolation model, and show that this process can be interpreted as a PCA.

2.1 The percolation PCA

Let $\mathcal{N} = \{n_1, \dots, n_m\}$ be a given subset of $\mathbb{Z}^d$, and consider the PCA Φ_p on $\{0,1\}^{\mathbb{Z}^d}$ defined by the local function:

$$\varphi_p(a_1, \dots, a_m) = p\, \delta_{\max\{a_1,\dots,a_m\}} + (1-p)\, \delta_0,$$

meaning that if a cell has at least a 1 in its neighbourhood, it becomes a 1 with probability p, and otherwise, it takes state 0. This PCA, which is also referred to as *Stavskaya PCA* for $d = 1$ and $\mathcal{N} = \{0,1\}$, can be seen as a dynamical version of directed percolation of neighbourhood $\mathcal{N}$. Assume that all sites are in state 1 in the initial configuration of the PCA. We can then couple the space-time diagram of Stavskaya PCA with a directed percolation process, in such a way that cell k is in state 1 at time t for the PCA if and only if in the percolation process, there is a directed path from some site $(l,0)$ to the site (k,t), along which all sites are in state 1. As a consequence, the PCA has two possible behaviour depending on whether p is smaller or larger than the percolation threshold $p_c(\mathcal{N})$ of the directed percolation model of neighbourhood $\mathcal{N}$. If $p < p_c(\mathcal{N})$, then whatever the initial configuration is, the state 1 dies out and the PCA converges to the configuration containing only the state 0, while if $p > p_c(\mathcal{N})$, from an initial configuration that contains cells in state 1, the state 1 has a positive survival probability.

2.2 Ergodicity criterion

We will now present a criterion of ergodicity that is based on a coupling of the trajectories from all the possible initial configurations, and on a comparison with directed percolation. Intuitively, a PCA is ergodic if it "forgets" its initial condition. In some cases, it is possible to prove the ergodicity in a constructive way, by making evolve simultaneously the trajectories from different initial conditions, using a common source of randomness, and showing that the evolutions of all these trajectories are asymptotically the same.
The *envelope PCA* allows one to systematize this idea of *coupling*. Instead of running the PCA from different initial configurations, we define a new PCA on an extended alphabet, containing a symbol `?` representing sites whose values are not known (i.e. which may differ between the different copies) and we run it from a single initial configuration containing only the symbol `?`. Each time we are able to make the different copies match on a cell, the symbol `?` is replaced by the state $q \in S$ on which the different copies agree. An evolution of the envelope PCA thus encodes a coupling of different copies of the original PCA, with a symbol `?` denoting sites where the copies disagree. If the density of symbols `?` converges to 0 when time goes to infinity, it means that the PCA is ergodic.

The envelope PCA was introduced in [3] as a tool to prove the ergodicity of a PCA and to generate perfect samples from its unique invariant distribution. The idea of the envelope PCA is reminiscent of the minorant PCA introduced by Toom et al. [22, Chap. 3], which can be used in a more or less similar fashion to prove ergodicity in the high-noise regime, and similar ideas have been pursued by others [8].

To simplify the presentation, let us assume that Φ is a PCA defined on a binary symbol set $S = \{0, 1\}$, and let $\tilde{S} = \{0, 1, \texttt{?}\}$. We define a partial order on $\tilde{S}$ by $0 \prec \texttt{?} \succ 1$. The *envelope PCA* $\tilde{\Phi}$ of Φ is the PCA of neighbourhood $\mathcal{N}$ and local function $\tilde{\varphi} : \tilde{S}^m \to \mathcal{M}(\tilde{S})$, defined for $q \in S$ by:

$$\tilde{\varphi}(y_1, \dots, y_m)(q) = \min\{\varphi(x_1, \dots, x_m)(q) : x_1 \preceq y_1, \dots, x_m \preceq y_m\},$$

where in the expression above, $x_1, \dots, x_m$ are taken in S. The probability of a transition to the symbol `?` is then given by:

$$\tilde{\varphi}(y_1, \dots, y_m)(\texttt{?}) = 1 - \tilde{\varphi}(y_1, \dots, y_m)(0) - \tilde{\varphi}(y_1, \dots, y_m)(1).$$

From a configuration $y \in \tilde{S}^{\mathbb{Z}^d}$, cell k is thus updated by the symbol $q \in S$ with the minimum of the probabilities of transition to the symbol q for Φ, taken over all the values of the neighbourhood of cell k that are compatible with the unknown cells of y. With the remaining probability, the cell is updated by a `?`.

In particular, in the evolution of the envelope PCA, at each time step, a cell is updated by the symbol `?` only if it has at least one neighbour in state `?`, and in that case, it becomes a `?` with probability at most:

$$p_{\texttt{?}} = 1 - \min_{x_1, \dots, x_m \in S} \varphi(x_1, \dots, x_m)(0) - \min_{x_1, \dots, x_m \in S} \varphi(x_1, \dots, x_m)(1).$$

This quantity measures how much the probability transitions depend on the value of the neighbourhood, and we have the following result [3].

Theorem 3. *Let $p_c(\mathcal{N})$ be the critical value of the two-dimensional directed site percolation of neighbourhood $\mathcal{N}$. If $p_? < p_c(\mathcal{N})$, then $\tilde{\Phi}^t \delta_{?^{\mathbb{Z}^d}}[?] \xrightarrow[t\to+\infty]{} 0$, so that the PCA Φ is ergodic.*

Sketch of the proof. Let us consider the directed graph describing the dependences between cells in the space-time diagram of the PCA. It is exactly the graph of set of vertices $\mathbb{Z}^d \times \mathbb{N}$, and bonds:

$$\boldsymbol{E} = \{((k,t),(k-n_i,t+1)) : k \in \mathbb{Z}^d, t \in \mathbb{Z}, n_i \in \mathcal{N}\}.$$

By dominating the process of symbols ? in the space-time diagram of the envelope PCA by Stavskaya PCA of parameter $p_?$, one proves that if $p_? < p_c(\mathcal{N})$, then the symbols ? die out. The result follows. □

2.3 The general ergodicity problem

As already noted, Theorem 3 only provides a sufficient condition for ergodicity. First, there exist simple examples of PCA that can be proved to be ergodic, but for which the envelope PCA is not ergodic. It is the case for example of the noisy sum PCA, when the noise ε is small [3]. Second, there are PCA for which the envelope PCA is ergodic, but with $p_? > p_c(\mathcal{N})$, so that the criterion of Theorem 3 is of no help for proving the ergodicity, see Section 3.2. In fact, Theorem 3 proves the ergodicity in the *high-noise regime*, when the local rule of the PCA depends sufficiently weakly on the value of the neighbourhood. Outside this regime, ergodicity is often difficult to prove, even in cases where it appears clear from heuristics or simulations. In Ref. [19], the authors gather different techniques for proving the ergodicity of PCA that are the perturbation of a deterministic CA by a small (positive) noise, but apart from specific families of CA (nilpotent, permutive, gliders, CA with a spreading symbol, surjective, algebraic), the problem is still largely open.

3 The hardcore PCA: a combinatorial excursion on percolation

In this section, we study from different points of view the binary one-dimensional PCA of neighbourhood $\mathcal{N} = \{0, 1\}$ that is presented in Fig. 4. We first present the close relationship between this PCA and the *hardcore model* from statistical physics. Then, we show that this PCA is related to percolation in at least two ways: it has an invariant distribution that is related to the counting series of directed percolation clusters, and it also appears in the study of a combinatorial game on percolation configurations.

3.1 The hardcore model

The hardcore model is a model of statistical mechanics in which particles are allowed to be on the vertices of a graph, but with the constraint that no two

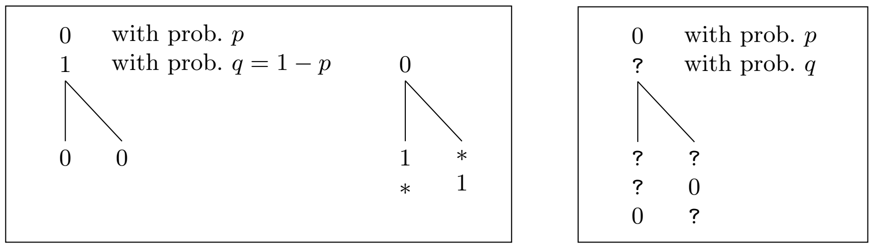


Fig. 4. Probability transitions of the hardcore PCA (left) and additional transitions of its envelope PCA (right). The symbol $*$ denotes an arbitrary symbol.

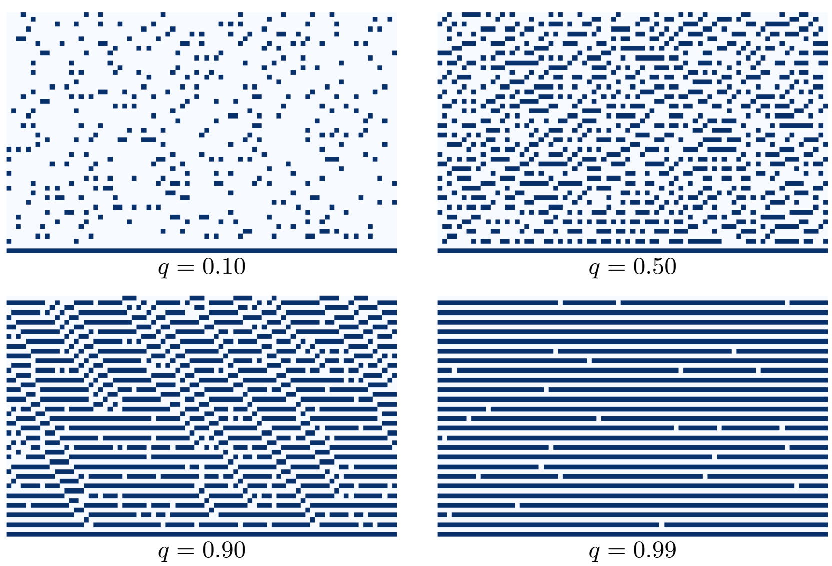


Fig. 5. Space-time diagrams of the hardcore PCA for the parameters $q = 0.10$, $q = 0.50$, $q = 0.90$ and $q = 0.99$, from the initial configuration “all 1” (time is going up).

particles may be adjacent. Formally, let $G=(V,E)$ be a finite graph. A configuration $\omega \in \{0,1\}^V$ is said to be a *hardcore configuration* (or an *independent set*) if for any $i,j \in V$ such that $\{i,j\} \in E$, $(\omega_i,\omega_j) \neq (1,1)$. We denote by $\mathcal{H}$ the set of hardcore configurations. Let $\lambda > 0$. The *hardcore measure on G with activity λ* is the probability distribution μ on $\{0,1\}^V$ defined by:

$$\forall \omega \in \{0,1\}^V, \quad \mu(\omega) = \frac{\lambda^{\sum_{i\in V}\omega_i}}{Z_\lambda}\,\mathbf{1}_{\omega\in\mathcal{H}},$$

where Z_λ is the *normalizing constant* (or *partition function*), defined by $Z_\lambda = \sum_{\omega\in\mathcal{H}} \lambda^{\sum_{i\in V}\omega_i}$. The distribution μ can also be characterized as follows: it is the probability measure under which the labels ω_i, for $i \in V$, are independent Bernoulli random variables of parameter $\lambda/(1+\lambda)$, conditioned on "having no adjacent 1's".

For an infinite graph G (countable, and locally finite), we can extend this definition by saying that a probability measure μ on $\{0,1\}^V$ is a hardcore measure on G with activity λ if the conditional distribution on a finite set A, given the configuration outside A, is just the distribution under which the vertices of A that are adjacent to a vertice of $V\setminus A$ in state 1 take value 0, and the distribution on the remaining set follows the (finite-case) hardcore measure with activity λ. It follows from standard argument of Gibbs measure theory that at least one such measure always exists. In the case of the lattice graph $(\mathbb{Z}^d, E_d)$, it is known that for $d=1$, for any activity $\lambda > 0$, there exists a unique hardcore measure, while a *phase transition* phenomenon occurs for $d \geq 2$: the hardcore measure is unique if λ is sufficiently small, but not for larger values of λ. In particular, when λ goes to infinity, there are two extremal Gibbs measure that concentrate asymptotically on the two odd and even "checkerboard" configurations.

Let us focus on the case of the one-dimensional lattice $\mathbb{Z}$. It can be seen that the (unique) hardcore measure μ of activity λ corresponds to the stationary Markov chain of transition matrix:

$$P_\lambda = \begin{pmatrix} p_{0,0} & p_{0,1} \\ p_{1,0} & p_{1,1} \end{pmatrix} = \begin{pmatrix} 1-x_\lambda & x_\lambda \\ 1 & 0 \end{pmatrix},$$

$$\text{with:} \qquad \frac{p_{0,1}p_{1,0}}{p_{0,1}p_{1,0}+p_{0,0}p_{0,0}} = \frac{x_\lambda}{x_\lambda+(1-x_\lambda)^2} = \frac{\lambda}{1+\lambda}.$$

Let X be a configuration distributed according to μ, and consider now the subfamily $X_{\text{even}} = (X_{2i})_{i\in\mathbb{Z}}$ of even cells, and the subfamily $X_{\text{odd}} = (X_{2i+1})_{i\in\mathbb{Z}}$ of odd cells. Then, X_{even} and X_{odd} are both distributed according to a same distribution μ_{half}, which is the stationary measure of the Markov chain of transition matrix P_λ^2. By definition of μ, if we update the state of a cell of X by a 1 with probability $\lambda/(1+\lambda)$ if it two neighbours are in state 0, and by a 0 otherwise, the resulting configuration still has distribution μ. This is also true if we update simultaneously all even (resp. odd) sites. As a consequence, the measure μ_{half}, seen as a measure on $\{0,1\}^{\mathbb{Z}}$, is an invariant distribution of the hardcore PCA of parameter $q=\lambda/(1+\lambda)$. This invariant distribution is *reversible*: if we first draw X_{even} according to μ_{half} and then update X_{odd} as above, the resulting configuration, which encodes the two consecutive configurations in the evolution of the PCA, has the same distribution as if we first draw the odd cells according to μ_{half} and then update the even cells using the PCA rule.

3.2 Ergodicity of the hardcore PCA

We have exhibited an invariant distribution of the one-dimensional hardcore PCA, having a Markovian structure. With the notations of Section 2.2, we have $p_{\texttt{?}} = q$. Thus, the criterion of Theorem 3 provides the ergodicity of the PCA only for small enough values of q. Furthermore, for large values of q, simulations can cast doubts on the asymptotic behaviour of the hardcore PCA, since starting from the configuration "all 1", we observe for a very long time an alternance between configurations with a very large majority of 1's and configurations with a very large majority of 0's (see Fig. 5), meaning that the system still remembers some information about the initial configuration. Nevertheless, one can prove the following.

Theorem 4. *For any $p \in (0,1)$, the hardcore PCA of parameter p is ergodic.*

There exist at least three ways to prove this result, each of them having some advantages and disavantages.

(i) **Connection with the hardcore PCA.** Thanks to a monotonicity argument, one can deduce the ergodicity of the PCA from the uniqueness of the Gibbs measure for the corresponding hardcore model, see the proof of Theorem 2 (ii) in Ref. [14].
(ii) **Weight system.** One can prove the ergodicity of the envelope PCA by introducing a suitable weight system that plays the role of a Lyapunov function, in order to prove that the density of the symbol `?` goes to zero, see Section 2.2. of Ref. [14].
(iii) **Decorrelated islands.** A proof of the ergodicity of the envelope PCA can also be made by studying the boundaries of islands of symbols `?` where the PCA is totally decorrelated from its initial condition [4].

The first proof strongly relies on the connection with the hardcore model, and allows one to recover the expression of the unique invariant measure of the hardcore PCA as the measure μ_{half} of "one in every two" cells of the Gibbs measure. It can be extended to other bipartite graphs for which the hardcore Gibbs measure is unique. In contrast, proofs (*i*) and (*ii*) do not require any knowledge about the hardcore model, but does not give any information about the expression of the unique invariant distribution of the hardcore PCA. However, even though the weighting system is quite specific to the local rule of this PCA, proof (*ii*) is still valid for a larger family of PCA [14]. The strategy of proof (*iii*) also allows us to handle hardcore PCA with two parameters, or defined on a neighbourhood of size 3, even if it is at the cost of many computations [5].

Actually, the ergodicity of the envelope PCA can also be seen as a consequence of the ergodicity of the hardcore PCA, using the monotonicity of the envelope PCA with respect to the order on configurations induced by the order on symbols defined by $0 \preceq \mathbf{?} \preceq 1$.

3.3 Directed animals

Directed animals are combinatorial objects related to directed percolation models. As shown by Dhar [6], enumerating directed animals according to the area on certain graphs is equivalent to solving a hard particle model on another graph. We will see that when the graph is the two-dimensional lattice, the counting series is then closely related to the unique invariant distribution of the hardcore PCA.

Consider the directed graph of set of vertices $\mathbb{Z}\times\mathbb{N}$, and edges $\boldsymbol{E} = \{((k,t),(k+v_i,t+1)) : k \in \mathbb{Z}, t \in \mathbb{N}, v_i \in \{0,1\}\}$. Let B be a non-empty finite subset of $\mathbb{Z}$. A *directed animal of base* B is a finite subset $\mathcal{A}$ of $\mathbb{Z} \times \mathbb{N}$ such that:

- $\mathcal{A} \cap (\mathbb{Z} \times \{0\}) = B \times \{0\}$,
- $\forall x \in \mathcal{A}$, $\exists n \in \mathbb{N}$, $\exists x_0 \dots, x_n \in \mathcal{A}$, $\begin{cases} x_0 \in B \times \{0\}, \quad x_n = x, \\ \forall i \in \{0,\dots,n-1\}, (x_i, x_{i+1}) \in \boldsymbol{E} \end{cases}$.

A *directed animal* is a directed animal of base $\{0\}$, see Fig. 6.

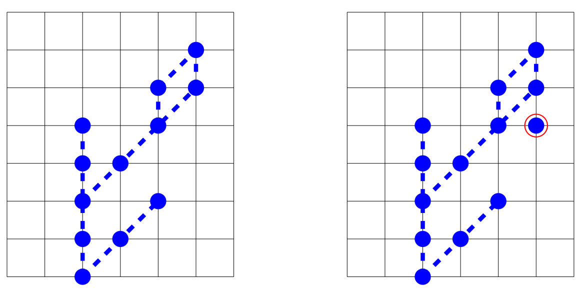


Fig. 6. A directed animal (left) and a set which is not a directed animal (right).

The *counting series* of directed animals of base B, respectively of directed animals, is the formal series defined by:

$$S_B(x) = \sum_{\mathcal{A}\text{: directed animal of base } B} x^{|\mathcal{A}|} \qquad (\text{resp. } S(x) = S_{\{0\}}(x)).$$

The coefficient of x^n in $S(x)$ thus gives the number of directed animals of size n. Removing the bottom line of a directed animal provides either the empty set or a new animal on the lines $\{1, 2, \dots\}$. This simple observation provides a recurrence relation on counting series:

$$S_B(x) = x^{|B|} \Big(\sum_{C \subset B+\mathcal{N}} S_C(x) \Big),$$

where $\mathcal{N} = \{0, 1\}$, and with the convention $S_\emptyset(x) = 1$.

Consider now a sequence of random variables $X = (X_i)_{i\in\mathbb{Z}}$ with values in $\{0, 1\}$, and let $Y = (Y_i)_{i\in\mathbb{Z}}$ be a realization of the image of $(X_i)_{i\in\mathbb{Z}}$ by the hardcore PCA of parameter q. By definition of the PCA, for a finite set $B \subset \mathbb{Z}$, we have:

$$\mathbb{P}\big(\forall i \in B, Y_i = 1\big) = \mathbb{P}\big(\forall i \in B + \mathcal{N}, X_i = 0\big) q^{|B|}.$$

It then follows from the inclusion-exclusion principle that:

$$\mathbb{P}\big(\forall i \in B + \mathcal{N}, X_i = 0\big) = \sum_{C \subset B+\mathcal{N}} (-1)^{|C|} \mathbb{P}\big(\forall i \in C,\ X_i = 1\big).$$

Assume that X is distributed according to an invariant distribution π of the PCA, and for $C \subset \mathbb{Z}$, let $T_C = \pi(\{\omega \in S^{\mathbb{Z}} : \forall i \in C, \omega_i = 1\})$.
Then, Y is also distributed according to π, and we obtain:

$$T_B = q^{|B|} \sum_{C \subset B+\mathcal{N}} (-1)^{|C|} T_C.$$

It follows that the family $(-1)^{|B|}T_B$, for finite subsets B of $\mathbb{Z}$, satisfies the same recursion as the family $S_B(-q)$. This provides an unexpected relation between these two models. Furthermore, since μ_{half} is one explicit invariant distribution of the hardcore PCA, it provides a candidate for the counting series of directed animals. The last step consists in arguing that the counting series indeed satisfies:

$$S_B(-q) = (-1)^{|B|} \mu_{\text{half}}(\{\omega \in S^{\mathbb{Z}} : \forall i \in B, \omega_i = 1\}).$$

This requires an argument since the recurrence relations may admit several families of solutions, with only one of them defining the counting series. The idea is to proceed in reverse direction: from the actual counting series, one can define a probability distribution π, that can be proved to be an invariant distribution of the hardcore PCA, and it is thus equal to μ_{half}, see Ref. [1,6,17] for details and extensions.

3.4 Playing with percolation

We now show that the hardcore PCA is also related to the study of a combinatorial game on percolation configurations. Let each cell of the square lattice $\mathbb{Z}^2$ be independently assigned one of the state *trap* with probability p, and *open* with probability $q = 1-p$, where $p \in [0, 1]$. Consider the following game: a token starts at the origin, and two players take turns to move, where a move consists of moving the token from its current cell x to either $x + (0, 1)$ or $x + (1, 0)$. A player who moves the token to a trap loses the game immediately. Otherwise (i.e. if the destination cell is open), the game continues with the other player's turn. The entire random assignment of traps and open cells to $\mathbb{Z}^2$ (which we call the *percolation configuration*) is known to both players at all times. We call this game the *percolation game* on $\mathbb{Z}^2$.

If $q \leq p_c$, where p_c is the critical probability for directed site percolation with neighbourhood $\mathcal{N} = \{0, 1\}$, then, with probability 1, only finitely many cells can be reached from the origin along directed paths of open cells, and so the game must end in finite time. In particular, one or other player must have a winning strategy, where a *strategy* for one or other player is a map that assigns a legal move (where one exists), to each cell, and a *winning* strategy is one that results in a win for that player, whatever strategy the other player uses. Suppose on the other hand that $q > p_c$, is there now a positive probability that neither player has a winning strategy? In that case, we say that the game is a *draw*, with the interpretation that it continues forever with the best play. Note that when $p = 0$, the game is clearly always a draw.

As we will see, the outcome (first-player win, first-player loss, draw) of the game started from each cell can be interpreted in terms of the evolution of the hardcore PCA, where the state of the PCA at a given time relates to the outcomes associated to the cells on a given Northwest-Southeast diagonal of $\mathbb{Z}^2$. Using the ergodicity of the hardcore PCA, one can prove that as soon as $p > 0$, the probability that the game is a draw is equal to 0, and the Markovian description of the invariant distribution then permits an explicit description of the distribution of game outcomes along a diagonal [14].

Theorem 5. *For any $p \in (0, 1)$, there is almost surely no draws, and the probability that the first player wins the game (conditional on the origin being open), is equal to:*

$$\frac{1 - 2p + \sqrt{\frac{p}{4-3p}}}{2(1-p)}. \tag{1}$$

This probability is greater than 1/2 if and only if $p \in (0, 1/3)$, and its maximum value is $4 - 2\sqrt{3} = 0.5358...$, attained at $p = (2 - \sqrt{3})/3 = 0.0893...$.

Sketch of the proof. Suppose x is an open site of $\mathbb{Z}^2$. Let $\eta(x)$ be W, L or D according to whether the game started with the token at x is a win for the first player, a loss for the first player, or a draw, respectively (recall that we assume optimal play, with the players able to see entire percolation configuration when deciding on their strategies). If x is a trap, it is convenient to set $\eta(x) = \mathrm{W}$ (we adopt the convention that if the game starts at a trap, then it is a win for the first player).

Let $\mathrm{Out}(x) = \{x + e_1, x + e_2\}$ be the set of cells to which the token can move from x. By considering the first move, we have the following recursion for the status of the cells:

$$x \text{ a trap } \Rightarrow \eta(x) = \mathrm{W},$$
$$x \text{ open } \Rightarrow \eta(x) = \begin{cases} \mathrm{L} & \text{if } \eta(y) = \mathrm{W} \text{ for all } y \in \mathrm{Out}(x), \\ \mathrm{W} & \text{if } \eta(y) = \mathrm{L} \text{ for some } y \in \mathrm{Out}(x), \\ \mathrm{D} & \text{otherwise.} \end{cases} \tag{2}$$

For $k \in \mathbb{Z}$, let S_k be the set $\{x = (x_1, x_2) \in \mathbb{Z}^2 : x_1 + x_2 = k\}$, a NW-SE diagonal of $\mathbb{Z}^2$. The recursion (2) gives us the values $(\eta(x))_{x \in S_k}$ in terms of the values $(\eta(x))_{x \in S_{k+1}}$ together with the information about which sites in S_k are traps. Via this recursion, we can regard the configurations on successive diagonals S_k, as k decreases, as successive states of a one-dimensional PCA. Let us introduce the following recoding:

$$\mathrm{W} = 0, \quad \mathrm{L} = 1, \quad \mathrm{D} = \texttt{?}.$$

Then, one can check that the PCA evolves as follows: given the values for cells in S_{k+1}, each value $\eta(x)$ for $x \in S_k$ is derived independently using the values $\eta(x + e_1)$ and $\eta(x + e_2)$, according to a local rule which is exactly the same as for the envelope of the hardcore PCA, see Fig. 4.

Let us fix some integer $N \geq 0$, and assume that on S_N, the configuration η is such that $\forall x \in S_k, \eta(x) = \texttt{?}$. Then, if we iterate the PCA starting from this configuration, from diagonal S_N to S_0, the probability that the origin $O = (0, 0)$ is in state 0 (resp. 1) can be interpreted as the probability that the first (respectively second) player can force a win within at most N moves of the game. In particular, if the probability that the origin is in state ? goes to 0 when N goes to infinity, it means that the game is almost surely not a draw. But as shown in Section 3.2, we already know that starting from the configuration $\delta_{\texttt{?}^{\mathbb{Z}}}$, the density of symbols ? goes to 0. This concludes the proof that there are no draws, and the description of the unique invariant measure of the hardcore PCA provides the expression of the winning probability. □

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