Ergodicity of the hard-core PCA with a random walk method

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Abstract. The hard-core probabilistic cellular automaton has attracted a renewed interest in the last few years, thanks to its connection with the study of a combinatorial game on percolation configurations. We provide an alternative proof for the ergodicity of this PCA for a neighbourhood of size 2 and 3, using the notion of decorrelated islands introduced by Casse in 2023, together with some new ideas. This shortens the previous proofs and provides a more intuitive and unified approach.

Keywords: probabilistic cellular automata \cdot ergodicity \cdot percolation \cdot hard-core model

1 Percolation games and PCA

Percolation games. Let us present the following percolation game on \mathbb{Z}^2 , as in Ref. [5]. Consider two non-negative reals $\epsilon_0, \epsilon_1 \in [0, 1]$ such that $0 \leq \epsilon_0 + \epsilon_1 \leq 1$. For each site of \mathbb{Z}^2 we assign independently one of the 3 following states:

- trap with probability ϵ_1 ;
- target with probability ϵ_0 ;
- open with probability $r = 1 \epsilon_0 \epsilon_1$.

This defines the random board on which two players compete. Fix an integer $n \ge 2$. The rule of the game is the following. At time 0, a token is placed at the starting position (0,0), and then, the two players move it alternatively, from its current position (i,j) on the board to a site in the set $\operatorname{Out}(i,j) = \{(i+k,j+1) : k \in \{0,1,\ldots,n-1\}\}$, see Figure 1. If the current player moves the token to a trap, that player loses the game immediately; if it moves it to a target, that player wins the game immediately; otherwise (i.e. if the destination site is open), the game continues with the other player's turn.

We can then ask the following question: with the best strategy for each player, what is the probability that the game never ends, so that there is a draw? The answer is 0 for the cases n = 2 [5] and n = 3 [1] when $0 < \epsilon_0 + \epsilon_1 \le 1$. In this article, we give alternative proofs of these two results, using ideas developed by Casse [3]. For $n \ge 4$, the question remains open.

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Fig. 1: The possible moves in the percolation game, for n = 2 on the left, and n = 3 on the right

Associated PCA. To study the percolation game, we can associate to each site the outcome (win, lose or draw) of the current player (the one who moves out the token from this site) when both players play perfectly. If we know the outcomes of the sites of the set Out(i, j), it allows to find out the outcome of site (i, j). This leads to the introduction of a probabilistic cellular automaton (PCA) having a neighborhood of size n, with three states: 0 for the win, 1 for the lose and ? for the draw. Note that a trap can be interpreted as a win site, since if a player moves the token to a trap, the next player (the one who should have moved the token out from the trap site) wins; while a target can be interpreted as a lose site.

This PCA, denoted \tilde{H}_n later in this article, defines a Markov chain $(X_i(t) : i \in \mathbb{Z})_{t>0}$ on $\{0, 1, ?\}^{\mathbb{Z}}$, whose transition probabilities satisfy:

$$\mathbb{P}(X_i(t+1) = a \mid \forall k \in \{0, \dots, n-1\}, X_{i+k}(t) = 0) = \begin{cases} \epsilon_0 & \text{if } a = 0; \\ 1 - \epsilon_0 & \text{if } a = 1; \\ 0 & \text{if } a = 2. \end{cases}$$

$$\mathbb{P}(X_i(t+1) = a \mid \exists k \in \{0, \dots, n-1\}, X_{i+k}(t) = 1) = \begin{cases} 1 - \epsilon_1 & \text{if } a = 0; \\ \epsilon_1 & \text{if } a = 1; \\ 0 & \text{if } a = 2. \end{cases}$$

$$\mathbb{P}\left(X_{i}(t+1) = a \mid (X_{i}(t), \dots, X_{i+n-1}(t)) \in \{0, ?\}^{n} \setminus \{(0, \dots, 0)\}\right) = \begin{cases} \epsilon_{0} & \text{if } a = 0; \\ \epsilon_{1} & \text{if } a = 1; \\ r & \text{if } a = 2; \end{cases}$$

These transitions are illustrated on Figure 2 for n = 2.

A PCA is *ergodic* if there is a unique probability measure μ_{∞} such that the Markov chain Z(t) of the PCA verifies: for any $\mu_0 \in \{0,1\}^{\mathbb{Z}}$ such that the initial configuration Z(0) is μ_0 , the law of Z(t) weakly converges to μ_{∞} . Here, the ergodicity of \tilde{H}_n means that the status of site (0,0) does not depend on



Fig. 2: Transition probability of the PCA H_2 .

the assignment of traps and targets that are sufficiently far away on the board. Observe that if for all $i \in \mathbb{Z}$, $X_i(0) \in \{0, 1\}$, then for all $i \in \mathbb{Z}$, $t \ge 0$, $X_i(t) \ne ?$. In particular, if the PCA \tilde{H}_n is ergodic, then its unique invariant measure does not contain the state ?, meaning that the probability of draw is null.

Theorem 1. For $n \in \{2,3\}$ and $(\epsilon_0, \epsilon_1) \in [0, \frac{1}{2}]^2 \setminus \{(0,0)\}$, the PCA \tilde{H}_n is ergodic, and so there is almost surely no draw in the percolation game.

The result above was first proved for n = 2 using weight functions [5]. The method was then adapted for n = 3, with highly intricate computations [1]. The aim of the present article is to provide another shorter proof, based on the method of decorrelated islands [3].

Note that the PCA H_n can be seen as the envelope PCA of the hard-core PCA H_n . By definition, it is the binary PCA whose associated Markov chain $X(t) = (X_i(t) : i \in \mathbb{Z}) \in \{0, 1\}^{\mathbb{Z}}$ satisfies:

$$\mathbb{P}(X_i(t+1) = a \mid \forall k \in \{0, \dots, n-1\} X_{i+k}(t) = 0) = \begin{cases} \epsilon_0 & \text{if } a = 0; \\ 1 - \epsilon_0 & \text{if } a = 1. \end{cases}$$

$$\mathbb{P}\left(X_{i}(t+1)=a \mid \exists k \in \{0,\dots,n-1\} X_{i+k}(t)=1\right) = \begin{cases} 1-\epsilon_{1} & \text{if } a=0;\\ \epsilon_{1} & \text{if } a=1. \end{cases}$$

The transitions are illustrated on Figure 3 for n = 3. The PCA H_n corresponds to the restriction of \tilde{H}_n to configurations that do not contain the state ?. Envelope PCA are a practical tool to prove the ergodicity of PCA in the high noise regime, using the fact that the ergodicity of the envelope PCA implies the ergodicity of the associated PCA [2].

Finally, let us also mention that H_n can be seen as a CA with *double errors*: it is a PCA obtained from a deterministic CA, on which we add an error depending on the expected state (ϵ_0 if we expect a 1 and ϵ_1 if we expect a 0).

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Fig. 3: Transition probability of the hard-core PCA H_3 , where dots mean that there can be either a 0 or a 1

Ergodicity of PCA. The ergodicity of PCA has been studied using various methods (coupling, entropy, Fourier analysis, weight functions ...) and still raises many questions, see [8,6,7] and the references therein. The hard-core PCA has attracted particular interest, since in addition to its connection to percolation games, it has long been known for its connection with statistical physics and with the enumeration of directed animals. For $\epsilon_1 = 0$, H_n has a Markovian invariant measure and its ergodicity is easier to prove [5], but for other values of the parameters, no general method is known.

Outline of the article. In Section 2, we present the sketch of our proof of Theorem 1. In Section 2.1, we recall the notion of decorrelated islands introduced by Casse [3], then in Section 2.2, we explain the new idea that allows to handle the case of the hard-core PCA. In Section 3, we prove Theorem 1 for n = 2. This requires some computations that we present in details. In Section 4, we prove Theorem 1 when n = 3, using the same method.

2 Sketch of the proof

2.1 Decorrelated islands method

Let us consider the evolution of the PCA \tilde{H}_n , from the initial configuration $\mathbb{C}^{\mathbb{Z}}$.

- With probability $(\epsilon_0 + \epsilon_1)^k$, the k consecutive cells of indices in $0, \ldots, k-1$ go from state ? to state 0 or 1, i.e. for all $i \in \{0, \ldots, k-1\}, X_i(1) \in \{0, 1\}$. We call such a sequence of consecutive cells a *decorrelated island*, as the states of these cells do not depend on the initial configuration.
- Our goal is to study the evolution of the size of a decorrelated island, that is a (maximal) set of consecutive cells where the symbol ? does not appear. With probability 1, there exists a time $t_0 \ge 0$ when such a decorrelated island is created. For $t \ge t_0$, we denote by i_t and j_t the positions of the left and right boundary of the decorrelated island, respectively. So, cells between i_t and j_t are in state 0 or 1, while $i_t - 1$ and $j_t + 1$ are in state ?.

• If $j_t - i_t \rightarrow_{t \rightarrow \infty} \infty$ with positive probability, this means that the island grows, so that the symbols ? progressively disappear, implying that the PCA is ergodic.

To study the behaviour of the left boundary i_t , we only remember the states of the first m cells of the island, given by $f_t = (X_{i_t}, X_{1+i_t}, \ldots, X_{m-1+i_t})$, with a well-chosen value of m, selected according to the size of the neighborhood n. We proceed in a similar way for the right boundary j_t . In the following, we will set m = 2 when n = 2, and m = 4 when n = 3.

This method, applied with m = 1, gives a sufficient ergodicity criterion for PCA with binary alphabet and two-size neighborhood [3]. This allows to handle the case of 12 out of 16 deterministic cellular automata with error $\epsilon > 0$. By taking m = 2, it also gives a new proof of the ergodicity of the PCA \tilde{H}_2 (and so of H_2) when $\epsilon_0 = \epsilon_1 > 0$.

2.2 The main improvement of the method

In this article, we improve the method developed by Casse [3] to give a new and shorter proof of Theorem 1. The improvement consists in considering one or two time steps at once, according to the value of the states of the boundary, and not just one time step in every situation.

First, observe that in order to prove that $\mathbb{P}(j_t - i_t \to \infty)$ is positive, since the size of negative steps is bounded (by 1), it is enough to show that the mean asymptotic drift

$$D = \lim_{t \to \infty} \mathbb{E}\left[J_t\right] - \lim_{t \to \infty} \mathbb{E}\left[I_t\right]$$

of $(j_t - i_t)_{t \ge t_0}$ is positive, where J_t is the increment of (j_t) , i.e. $J_t = j_{t+1} - j_t$ and I_t the one of (i_t) . The following remark in fact allows one to restrict ourselves to the study of the right boundary.

Remark 1. Let us denote by $R = \lim_{t\to\infty} \mathbb{E} [J_t]$ the mean asymptotic drift of the right boundary. Using the symmetry of the transition rule of \tilde{H}_n with respect to the left-right symmetry, while taking into account the fact that the neighborhood is not centered, it follows that D = 2R + (n-1). Consequently, in the rest of the article, we focus only on the behaviour of the right boundary.

The innovation of the present work is to study the law of J_t and the one of $J_t + J_{t+1}$ according to the values of the states of the boundary f_t . This is summed up in the following lemma.

Lemma 1. Consider a homogeneous Markov chain (j_t, f_t) on $\mathbb{Z} \times F$, where F is a finite set, and assume that the increment $J_t = j_{t+1} - j_t$ depends only on f_t . Precisely, assume that for any $f \in F$, for any $k \in \mathbb{Z}$, and for any $t \ge 0$,

$$\mathbb{E}[J_t | f_t = f, j_t = k] = \mathbb{E}[j_1 | f_0 = f, j_0 = 0].$$

Then, if the limit $R = \lim_{t\to\infty} \mathbb{E}[J_t]$ exists and is finite, it satisfies:

$$R \ge \min_{f \in F} \max\left(\mathbb{E}\left[J_t | f_t = f\right], \frac{1}{2} \mathbb{E}\left[J_t + J_{t+1} | f_t = f\right] \right).$$
(1)

Proof. First, we split the set F into two disjoint parts:

$$F_1 = \left\{ f \in F : \mathbb{E} \left[J_0 | f_0 = f \right] \ge \frac{1}{2} \mathbb{E} \left[J_0 + J_1 | f_0 = f \right] \right\} \text{ and } F_2 = F \setminus F_1.$$

The set F_1 represents the states for which the mean increment is larger when we consider a single time step compared to two time steps, and F_2 those for which it is the reverse.

We now define another Markov chain $(\hat{j}_t, \hat{f}_t, w_t)_{t \ge 0}$ as follows:

- $(\hat{j}_0, \hat{f}_0, w_0) = (j_0, f_0, \mathbb{1}_{f_0 \in F_2}),$
- if $w_t = 0$, then $\hat{j}_{t+1} = \hat{j}_{t+1}$, $\hat{f}_{t+1} = f_{t+1}$ and $w_{t+1} = \mathbb{1}_{f_{t+1} \in F_2}$,
- if $w_t = 1$, then $\hat{j}_{t+1} = j_t$, $\hat{f}_{t+1} = f_t$ and $w_{t+1} = 0$.

As a consequence:

$$\hat{J}_t = \hat{j}_{t+1} - \hat{j}_t = J_t \text{ when } w_t = 0 \text{ and } f_t \in F_1;$$
$$\hat{J}_t = 0 \text{ when } w_t = 1 \text{ and } f_t \in F_2;$$
$$\hat{J}_t = J_{t-1} + J_t \text{ when } w_t = 0 \text{ and } f_t \in F_2.$$

So, compared to $(j_t, f_t)_{t\geq 0}$, the new Markov chain moves one step forward when \hat{f}_t is in F_1 , while when it is in F_2 , the state does not change at the first time step (when $w_t = 1$) and then moves of the equivalent of two steps at a time (when $w_t = 0$). The variable w_t can thus be thought as a waiting time.

Let us now consider the value $\hat{R} = \lim_{t\to\infty} \mathbb{E}\left[\hat{J}_t\right]$. We show that $\hat{R} \geq M$, where M is the term on the right-hand side of (1). We have

$$\hat{R} = \sum_{f \in F_1} \tau(f, 0) \mathbb{E} \left[\hat{J}_t | \hat{f}_t = f, w_t = 0 \right] + \sum_{f \in F_2} \left(\tau(f, 1) \times 0 + \tau(f, 0) \mathbb{E} \left[\hat{J}_t | \hat{f}_t = f, w_t = 0 \right] \right)$$

where τ is the invariant measure of the Markov chain $(\hat{f}_t, w_t)_{t\geq 0}$, which exists since F is finite.

By definition of $(\hat{j}_t, \hat{f}_t, w_t)_{t \ge 0}$, we have for any $f \in F_1$,

$$\mathbb{E}\left[\hat{J}_t|\hat{f}_t = f, w_t = 0\right] = \max\left(\mathbb{E}\left[J_t|f_t = f\right], \frac{1}{2}\mathbb{E}\left[J_t + J_{t+1}|f_t = f\right]\right) \ge M$$

and for any $f \in F_2$,

$$\mathbb{E}\left[\hat{J}_t|\hat{f}_t = f, w_t = 0\right] = 2\max\left(\mathbb{E}\left[J_t|f_t = f\right], \frac{1}{2}\mathbb{E}\left[J_t + J_{t+1}|f_t = f\right]\right) \ge 2M.$$

By construction of the new Markov chain, for any $f \in F_2$, $\tau(f, 0) = \tau(f, 1)$. So we obtain $\hat{R} \ge M \sum_{(f,w) \in (F_1 \times \{0\}) \cup (F_2 \times \{0,1\})} \tau(f,w) = M$.

To conclude, we only need to prove that $R = \hat{R}$. We recall that the equality $(j_t, f_t) = (\hat{j}_t, \hat{f}_t)$ holds at least once in two, and by the ergodic theorem,

$$\hat{R} = \lim_{t \to \infty} \mathbb{E}\left[\hat{J}_t\right] = \lim_{t \to \infty} \frac{\mathbb{E}\left[\hat{j}_t - j_0\right]}{t} \text{ and } R = \lim_{t \to \infty} \mathbb{E}\left[J_t\right] = \lim_{t \to \infty} \frac{\mathbb{E}\left[j_t - j_0\right]}{t}.$$

So, we have $\hat{R} = R$.

Using Remark 1 together with Lemma 1, we can then prove Theorem 1 by just showing that, for any $n \in \{2, 3\}$ and for every $t \ge t_0$,

$$R \ge \min_{f \in \{0,1\}^m} \left(\max\left(\mathbb{E}\left[J_t | f_t = f\right], \frac{1}{2} \mathbb{E}\left[J_t + J_{t+1} | f_t = f\right] \right) \right) > -\frac{n-1}{2}.$$
 (2)

Remark 2. We will actually prove the previous inequality not for (j_t) but for some modification (\tilde{j}_t) , introduced in Sections 3.1 for n = 2 and in 4.1 for n = 3.

3 Proof of Theorem 1 when n = 2

3.1 Modified boundary and star state

To give a formal proof of Theorem 1, we now introduce two additional notions.

First, as mentioned in Remark 2, we define a slight modification of the position j_t , that depends on the states of the boundaries. To do so, let us consider a decorrelated island $(X_{i_t}, \ldots, X_{j_t})$ not reduced to a singleton at time t. To study the behaviour of its right boundary, we introduce the modified position \tilde{j}_t defined by

$$\tilde{j}_t = \begin{cases} j_t & \text{if } (X_{-1+j_t}, X_{j_t}) \in \{(0, 1), (1, 1)\};\\ j_t - \frac{1}{2} & \text{if } (X_{-1+j_t}, X_{j_t}) = (0, 0);\\ j_t - 1 & \text{if } (X_{-1+j_t}, X_{j_t}) = (1, 0). \end{cases}$$

As a result of this technical change, if $\epsilon_0 = \epsilon_1 = 0$ (error-free regime), then the deterministic drift satisfies $\tilde{J}_t = -1/2$ for any t, whereas J_t oscillates between 0 and -1. The purpose of this modification is therefore only to simplify the analysis. Note that the cells we will consider at time t in the right boundary will still be given by $f_t = (X_{-1+j_t}, X_{j_t})$. The modification \tilde{i}_t of i_t is defined analogously, using the left-right symmetry of the transition rule.

Second, we introduce a star state, that we denote by *. This state will be used to encode the state of a cell of the decorrelated island whose exact value (0 or 1) is not remembered. Indeed, if we retain all the information about the island, the computation is too complex. The state * will thus enable us to establish some bounds for the behaviour of the boundaries. This is different from the state ?, which represents a cell whose value is completely unknown.

3.2 Transition probabilities

In this section, we study the behaviour of the Markov chain (\tilde{j}_t, f_t) where we recall that $f_t = (X_{-1+j_t}, X_{j_t})$. Our goal is to prove that

$$\mathbb{E}\left[\tilde{J}_t | f_t \in \{(0,1), (1,0), (1,1)\}\right] > -\frac{1}{2} \text{ and } \mathbb{E}\left[\tilde{J}_t + \tilde{J}_{t+1} | f_t = (0,0)\right] > -1, (3)$$

which implies Equation (2) when n = 2, and so Theorem 1 for n = 2.

In the following, we assume that $j_t - i_t \geq 5$. It is not a loss of generality to achieve the result, and it allows to avoid any problem of dependency between $(\tilde{i}_{t+1}, (X_{i_{t+1}}, X_{1+i_{t+1}}))$ and $(\tilde{j}_{t+1}, (X_{-1+j_{t+1}}, X_{j_{t+1}}))$.

Case $f_t \in \{(0, 1), (1, 0), (1, 1)\}$: For these three types of right boundaries, the transition probabilities are the same, as illustrated on Figure 4.

More precisely, if $\tilde{j}_t = \tilde{j}$ and if $f_t = f$ belongs to $\{(0, 1), (1, 0), (1, 1)\}$, then

$$(\tilde{j} - 1, (1, 0)) \quad \text{w.p. } \epsilon_1 (1 - \epsilon_1) r \\ (\tilde{j} - \frac{1}{2}, (0, 0)) \quad \text{w.p. } (1 - \epsilon_1)^2 r \\ (\tilde{j}, (0, 1)) \quad \text{w.p. } (1 - \epsilon_1) \epsilon_1 r \\ (\tilde{j}, (1, 1)) \quad \text{w.p. } \epsilon_1^2 r \\ (\tilde{j}, (1, 0)) \quad \text{w.p. } \epsilon_1 \epsilon_0 r \\ (\tilde{j} + \frac{1}{2}, (0, 0)) \quad \text{w.p. } (1 - \epsilon_1) \epsilon_0 r \\ (\tilde{j} + 1, (0, 1)) \quad \text{w.p. } (1 - \epsilon_1) \epsilon_1 r \\ (\tilde{j} + 1, (1, 1)) \quad \text{w.p. } \epsilon_1^2 r \\ (\tilde{j} + k + 1, (1, 0)) \quad \text{w.p. } (\epsilon_0 + \epsilon_1)^k \epsilon_1 \epsilon_0 r \text{ for } k \ge 0 \\ (\tilde{j} + k + \frac{3}{2}, (0, 0)) \quad \text{w.p. } (\epsilon_0 + \epsilon_1)^k \epsilon_0 \epsilon_1 r \text{ for } k \ge 0 \\ (\tilde{j} + k + 2, (0, 1)) \quad \text{w.p. } (\epsilon_0 + \epsilon_1)^k \epsilon_0 \epsilon_1 r \text{ for } k \ge 0 \\ (\tilde{j} + k + 2, (1, 1)) \quad \text{w.p. } (\epsilon_0 + \epsilon_1)^k \epsilon_1^2 r \text{ for } k \ge 0 \\ (\tilde{j} + k + 2, (1, 1)) \quad \text{w.p. } (\epsilon_0 + \epsilon_1)^k \epsilon_1^2 r \text{ for } k \ge 0 \\ \end{array}$$

Using Equation (4), we can compute the mean increment in one time step for this type of boundary:

$$\mathbb{E}\left[\tilde{J}_{t}|f_{t} = \{(0,1),(1,0),(1,1)\}\right]$$

$$= -\epsilon_{1}(1-\epsilon_{1})r - \frac{1}{2}(1-\epsilon_{1})^{2}r + \frac{1}{2}\epsilon_{0}(1-\epsilon_{1})r + \dots + \sum_{k=0}^{\infty}(k+2)\epsilon_{1}^{2}(1-r)^{k}r$$

$$= -\frac{1}{2}r + \epsilon_{1}r + \frac{1}{2}\epsilon_{0}(1-\epsilon_{1})r + \frac{1}{2}\epsilon_{1}^{2}r + \frac{1}{2}\epsilon_{0}^{2} + \epsilon_{0}\epsilon_{1} + \epsilon_{1}^{2} + \frac{(1-r)^{2}}{r}$$

$$= -\frac{1}{2} + \underbrace{\frac{1}{2}\epsilon_{0} + \frac{1}{2}\epsilon_{1} + \epsilon_{1}r + \frac{1}{2}\epsilon_{0}(1-\epsilon_{1})r + \frac{1}{2}\epsilon_{1}^{2}r + \frac{1}{2}(1-r)^{2} + \frac{1}{2}\epsilon_{1}^{2} + \frac{(1-r)^{2}}{r} }{}_{>0}$$

$$(5)$$

This can be checked by hand or with the help of a computer algebra system.



Fig. 4: Transition probabilities of the different cells when the state f of the boundary belongs to $\{(0,1), (1,0), (1,1)\}$. The first three lines correspond to the three cases f = (1,0), f = (0,1) and f = (1,1). The boundary position j depends on the case, but the modified boundary \tilde{j} is always the same. The boxes provide the different possible states of the cells with their respective probability.

Case $f_t = (0, 0)$: For this type of boundary, we look at the increment in two time steps. The first step is given on Figure 5.

First step: As illustrated on Figure 5, if $\tilde{j}_t = \tilde{j}$ and if the state of the boundary satisfies $f_t = (0, 0)$, then

$$(\tilde{j}_{t+1}, f_{t+1}) = \begin{cases} \left(\geq \tilde{j} - \frac{3}{2}, (*, 0) \right) & \text{w.p. } \epsilon_0 r^2 \\ \left(\tilde{j} - \frac{3}{2}, (1, 0) \right) & \text{w.p. } \epsilon_1 \epsilon_0 r \\ \left(\tilde{j} - 1, (0, 0) \right) & \text{w.p. } \epsilon_0^2 r \\ \left(\tilde{j} - \frac{1}{2}, (0, 1) \right) & \text{w.p. } \epsilon_0 (1 - \epsilon_0) r \\ \left(\tilde{j} - \frac{1}{2}, (1, 1) \right) & \text{w.p. } \epsilon_1 (1 - \epsilon_0) r \\ \left(\tilde{j} - \frac{1}{2}, (*, 1) \right) & \text{w.p. } r(1 - \epsilon_0) r \\ \left(\tilde{j} - \frac{1}{2}, (1, 0) \right) & \text{w.p. } (1 - \epsilon_0) \epsilon_0 r \\ \left(\tilde{j} + \frac{1}{2}, (0, 1) \right) & \text{w.p. } \epsilon_0 \epsilon_1 r \\ \left(\tilde{j} + \frac{1}{2}, (1, 1) \right) & \text{w.p. } (1 - \epsilon_0) \epsilon_1 r \\ \left(\tilde{j} + k + \frac{1}{2}, (1, 0) \right) & \text{w.p. } (\epsilon_0 + \epsilon_1)^k \epsilon_0 r \\ \left(\tilde{j} + k + \frac{1}{2}, (0, 1) \right) & \text{w.p. } (\epsilon_0 + \epsilon_1)^k \epsilon_0 r \\ \left(\tilde{j} + k + \frac{3}{2}, (0, 1) \right) & \text{w.p. } (\epsilon_0 + \epsilon_1)^k \epsilon_0 \epsilon_1 r \\ \left(\tilde{j} + k + \frac{3}{2}, (1, 1) \right) & \text{w.p. } (\epsilon_0 + \epsilon_1)^k \epsilon_0 \epsilon_1 r \\ \left(\tilde{j} + k + \frac{3}{2}, (1, 1) \right) & \text{w.p. } (\epsilon_0 + \epsilon_1)^k \epsilon_0 \epsilon_1 r \\ \left(\tilde{j} + k + \frac{3}{2}, (1, 1) \right) & \text{w.p. } (\epsilon_0 + \epsilon_1)^k \epsilon_0 \epsilon_1 r \\ \left(\tilde{j} + k + \frac{3}{2}, (1, 1) \right) & \text{w.p. } (\epsilon_0 + \epsilon_1)^k \epsilon_0 \epsilon_1 r \\ \left(\tilde{j} + k + \frac{3}{2}, (1, 1) \right) & \text{w.p. } (\epsilon_0 + \epsilon_1)^k \epsilon_0 \epsilon_1 r \\ \left(\tilde{j} + k + \frac{3}{2}, (1, 1) \right) & \text{w.p. } (\epsilon_0 + \epsilon_1)^k \epsilon_0 \epsilon_1 r \\ \left(\tilde{j} + k + \frac{3}{2}, (1, 1) \right) & \text{w.p. } (\epsilon_0 + \epsilon_1)^k \epsilon_0 \epsilon_1 r \\ \left(\tilde{j} + k + \frac{3}{2}, (1, 1) \right) & \text{w.p. } (\epsilon_0 + \epsilon_1)^k \epsilon_0 \epsilon_1 r \\ \left(\tilde{j} + k + \frac{3}{2}, (1, 1) \right) & \text{w.p. } (\epsilon_0 + \epsilon_1)^k \epsilon_0 \epsilon_1 r \\ \left(\tilde{j} + k + \frac{3}{2}, (1, 1) \right) & \text{w.p. } (\epsilon_0 + \epsilon_1)^k \epsilon_0 \epsilon_1 r \\ \left(\tilde{j} + k + \frac{3}{2}, (1, 1) \right) & \text{w.p. } (\epsilon_0 + \epsilon_1)^k \epsilon_0 \epsilon_1 r \\ \left(\tilde{j} + k + \frac{3}{2}, (1, 1) \right) & \text{w.p. } (\epsilon_0 + \epsilon_1)^k \epsilon_0 \epsilon_1 r \\ \left(\tilde{j} + k + \frac{3}{2}, (1, 1) \right) & \text{w.p. } (\epsilon_0 + \epsilon_1)^k \epsilon_0 \epsilon_1 r \\ \left(\tilde{j} + k + \frac{3}{2}, (1, 1) \right) & \text{w.p. } (\epsilon_0 + \epsilon_1)^k \epsilon_0 \epsilon_1 r \\ \left(\tilde{j} + k + \frac{3}{2}, (1, 1) \right) & \text{w.p. } (\epsilon_0 + \epsilon_1)^k \epsilon_0 \epsilon_1 r \\ \left(\tilde{j} + k + \frac{3}{2}, (1, 1) \right) & \text{w.p. } (\epsilon_0 + \epsilon_1)^k \epsilon_0 \epsilon_1 r \\ \left(\tilde{j} + k + \frac{3}{2}, (1, 1) \right) & \text{w.p. } (\epsilon_0 + \epsilon_1)^k \epsilon_0 \epsilon_1 r \\ \left(\tilde{j} + k + \frac{3}{2}, (1, 1) \right) & \text{w.p. } (\epsilon_0 + \epsilon_1)^k \epsilon_1 r \\ \end{array} \right\}$$

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Fig. 5: Transition probabilities when the boundary is in state (0,0) and \tilde{j} indicates the position of the boundary with the modification considered.

Remark 3. In the first case, which corresponds to a transition to $f_{t+1} = (*, 0)$, the value of \tilde{j}_{t+1} can be equal to $\tilde{j} - 3/2$ if the boundary is in state (1, 0) or to $\tilde{j} - 1$ if it is in state (0, 0). As we want to bound by below the increment, we keep the lower bound $\tilde{j} - 3/2$ in that case.

Second step: As the second step depends only on the first step through the value of f_{t+1} , the transitions given in Equation (6) allow to deduce the mean increment for two time steps,

$$\mathbb{E}\left[\tilde{J}_{t} + \tilde{J}_{t+1}|f_{t} = (0,0)\right]$$

= $\mathbb{E}\left[\tilde{J}_{t}|f_{t} = (0,0)\right] + \mathbb{P}\left(f_{t+1} = (*,0)|f_{t} = (0,0)\right) \mathbb{E}\left[\tilde{J}_{t+1}|f_{t+1} = (*,0)\right]$
+ $\mathbb{P}\left(f_{t+1} = (0,0)|f_{t} = (0,0)\right) \mathbb{E}\left[\tilde{J}_{t+1}|f_{t+1} = (0,0)\right]$
+ $\mathbb{P}\left(f_{t+1} \in \{(0,1), (1,0), (1,1), (*,1)\}|f_{t} = (0,0)\right) \mathbb{E}\left[\tilde{J}_{t+1}|f_{t+1} = (0,1)\right].$ (7)

The probability transitions of f_{t+1} are given by Equation (6):

$$\mathbb{P}\left(f_{t+1} = (*,0)|f_t = (0,0)\right) = \epsilon_0 r^2, \\ \mathbb{P}\left(f_{t+1} = (0,0)|f_t = (0,0)\right) = 2\epsilon_0^2 r + \sum_{k=0}^{\infty} \epsilon_0^2 (1-r)^k r = 2\epsilon_0^2 r + \epsilon_0^2, \text{ and, so,} \\ \mathbb{P}\left(f_{t+1} \in \{(0,1), (1,0), (1,1), (*,1)\}|f_t = (0,0)\right) = 1 - \epsilon_0 r^2 - 2\epsilon_0^2 r - \epsilon_0^2.$$

Moreover, by Equation (6), the mean increment in one time step when the boundary is in state (0,0) verify

$$\mathbb{E}\left[\tilde{J}_t|f_t = (0,0)\right] \ge -\frac{1}{2}r^2 - 2\epsilon_0 r + \epsilon_0^2 r + \frac{1}{2}\epsilon_1^2 + \frac{(1-r)^2}{r}$$
(8)

and, when it is in state (*, 0) we have the following lower bound:

$$\mathbb{E}\left[\tilde{J}_{t}|f_{t} = (*,0)\right] \geq \min\left(\mathbb{E}\left[\tilde{J}_{t}|f_{t} = (0,0)\right], \mathbb{E}\left[\tilde{J}_{t}|f_{t} = (1,0)\right]\right)$$
$$\geq -\frac{1}{2}r^{2} - 2\epsilon_{0}r + \epsilon_{0}^{2}r + \frac{1}{2}\epsilon_{1}^{2} + \frac{(1-r)^{2}}{r}.$$
(9)

Indeed,

$$\mathbb{E}\left[\tilde{J}_t|f_t = (1,0)\right] - \left(-\frac{1}{2}r^2 - 2\epsilon_0 r + \epsilon_0^2 r + \frac{1}{2}\epsilon_1^2 + \frac{(1-r)^2}{r}\right)$$
$$= \frac{1}{2}(1-r)^2 + \frac{1}{2}\epsilon_1 r + \frac{1}{2}\epsilon_1^2 r + \frac{1}{2}\epsilon_0(1-\epsilon_1)r + \epsilon_0\left(\frac{3}{2} - \epsilon_0\right)r \ge 0.$$
(10)

Hence, as detailed in Appendix 6.1, we finally obtain that

$$\mathbb{E}\left[\tilde{J}_t + \tilde{J}_{t+1}|f_t = (0,0)\right] > -1.$$
(11)

Equations (5) and (11) imply Equation (2) when n = 2, and so Theorem 1 for n = 2.

4 Proof of Theorem 1 when n = 3

In this Section, we consider the PCA \tilde{H}_3 . The proof follows the same way as for \tilde{H}_2 . The main difference is that we now consider a boundary f_t of size 4. Also, the definition of the modified boundary \tilde{j}_t is slightly different.

4.1 Modified boundary

Let us consider a decorrelated island $(X_{i_t}, \ldots, X_{j_t})$. We now define the modified boundary position \tilde{j}_t by

$$\tilde{j}_t = \begin{cases} j_t & \text{if } X_{j_t} = 1, \\ j_t - 1 & \text{if } (X_{j_t - 2}, X_{j_t - 1}, X_{j_t}) = (0, 0, 0) \text{ or } (X_{j_t - 1}, X_{j_t}) = (1, 0), \\ j_t - 2 & \text{else, i.e. if } (X_{j_t - 2}, X_{j_t - 1}, X_{j_t}) = (1, 0, 0). \end{cases}$$

4.2 Transition probabilities

For n = 3, we consider a right boundary made of the states of the 4 rightmost cells of the island, $f_t = (X_{-3+j_t}, X_{-2+j_t}, X_{-1+j_t}, X_{j_t})$, and in the following, we assume that $j_t - i_t \ge 9$. It is not a loss of generality to achieve the result, and it allows to avoid any problem of dependency between (\tilde{j}_{t+1}, f_t) and its left counterpart.

We group the possibles states of the boundary into three subsets of configurations (two being reduced to a singleton): $\{(0,0,0,0)\}$, $\{(1,0,0,0)\}$ and

$$S_1 = \{(a_0, a_1, a_2, a_3) : a_0 \in \{0, 1\} \text{ and } \exists k \in \{1, 2, 3\}, a_k = 1\}.$$

The 14 configurations of this last set can be considered together, as their transition probabilities are the same. This is similar to the case n = 2, when (0, 1), (1, 0) and (1, 1) were grouped. **Case** $f_t \in S_1$: For this case, we can reduce the boundary to a length equal to 3, as the mean increment in one time step is enough. Hence, in the transition, we do not care about the value of $X_{j_{t+1}-3}$, as illustrated on Figure 6. If $\tilde{j}_t = \tilde{j}$ and $f_t = f \in S_1$, then, we have

$$\begin{split} & \left(\tilde{j}_{t+1}, (X_{j_{t+1}-2}, X_{j_{t+1}-1}, X_{j_{t+1}}) \right) = \\ & \left\{ \begin{aligned} & \left(\tilde{j} - 1 + k, (0, 0, 0) \right) & \text{w.p. } (1 - \epsilon_1)^{3-k} \epsilon_0^k r \text{ for } k \in \{0, 1, 2\}; \\ & \left(\tilde{j} + 2 + k, (0, 0, 0) \right) & \text{w.p. } (\epsilon_0 + \epsilon_1)^k \epsilon_0^3 r \text{ for } k \in \mathbb{N}; \\ & \left(\tilde{j} - l + k, \{0, 1\}^{2-l} \times \{1\} \times \{0\}^l \right) & \text{w.p. } \epsilon_1 (1 - \epsilon_1)^{l-k} \epsilon_0^k r \text{ for } l \in \{0, 1, 2\} \text{ and } 0 \le k \le l; \\ & \left(\tilde{j} - l + k, \{0, 1\}^{2-l} \times \{1\} \times \{0\}^l \right) & \text{w.p. } (\epsilon_0 + \epsilon_1)^{k-l-1} \epsilon_1 \epsilon_0^l r \text{ for } l \in \{0, 1, 2\} \text{ and } l + 1 \le k \le 2; \\ & \left(\tilde{j} + 3 - l + k, \{0, 1\}^{2-l} \times \{1\} \times \{0\}^l \right) & \text{w.p. } (\epsilon_0 + \epsilon_1)^{k+2-l} \epsilon_1 \epsilon_0^l r \text{ for } l \in \{0, 1, 2\} \text{ and } k \in \mathbb{N}; \end{aligned}$$

We then obtain the following mean increment:

$$\mathbb{E}\left[\tilde{J}_t | f_t \in S_1\right] = \sum_{k=0}^2 (-1+k)(1-\epsilon_1)^{3-k} \epsilon_0^k r + \dots$$

= $-1 + \epsilon_0 + 2\epsilon_1 + \epsilon_0^2 + 3\epsilon_1^2(1-\epsilon_1) + \epsilon_0\epsilon_1(1-\epsilon_0) + \epsilon_1^4 + \epsilon_0^3\epsilon_1 + \frac{(1-r)^3}{r} > -1.$ (12)

This can be checked by hand or with the help of a computer algebra system. Hence, we proved Equation (2) for any $f \in S_1$. For the two others cases (f = (0, 0, 0, 0) or f = (1, 0, 0, 0)), we have to consider two time steps. The case $f_t = (1, 0, 0, 0)$ is considered first.

Case $f_t = (1, 0, 0, 0)$: As illustrated on Figure 7, if $\tilde{j}_t = \tilde{j}$ and $f_t = (1, 0, 0, 0)$, then

$$\begin{split} &(\tilde{j}_{t+1},f_{t+1}) = \\ & \begin{cases} (\tilde{j}-2+k,(0,0,0,0)) & \text{w.p. } (1-\epsilon_1)^{3-k}\epsilon_0^{1+k}r \text{ for } k \in \{0,1,2,3\}; \\ (\tilde{j}-2+k,(1,0,0,0)) & \text{w.p. } \epsilon_1(1-\epsilon_1)^{2-k}\epsilon_0^{1+k}r \text{ for } k \in \{0,1,2\}; \\ (\tilde{j}+1,(1,0,0,0)) & \text{w.p. } (1-\epsilon_0)\epsilon_0^3r; \\ (\tilde{j}-1+k-l,\{0,1\}^{3-l}\times\{1\}\times\{0\}^l) & \text{w.p. } \epsilon_0^{k+1}\epsilon_1(1-\epsilon_1)^{l-(k+1)}r \text{ for } l \in \{1,2\} \text{ and } 0 \le k \le l-1; \\ (\tilde{j}-1,\{0,1\}^{3-l}\times\{1\}\times\{0\}^l) & \text{w.p. } (1-\epsilon_0)\epsilon_0^lr \text{ for } l \in \{0,1,2\}; \\ (\tilde{j}-1+k-l,\{0,1\}^{3-l}\times\{1\}\times\{0\}^l) & \text{w.p. } (\epsilon_0+\epsilon_1)^{k-(l+1)}\epsilon_1\epsilon_0^lr \text{ for } l \in \{0,1,2\} \text{ and } l+1 \le k \le 3; \\ (\tilde{j}+2+k,(0,0,0,0)) & \text{w.p. } (\epsilon_0+\epsilon_1)^k\epsilon_0^ar \text{ for } k \in \mathbb{N}; \\ (\tilde{j}+3+k-l,\{0,1\}^{3-l}\times\{1\}\times\{0\}^l) & \text{w.p. } (\epsilon_0+\epsilon_1)^{3+k-l}\epsilon_1\epsilon_0^lr \text{ for } l \in \{0,1,2\} \text{ and } k \in \mathbb{N}; \\ (\tilde{j}+3+k-l,\{0,1\}^{3-l}\times\{1\}\times\{0\}^l) & \text{w.p. } (\epsilon_0+\epsilon_1)^{3+k-l}\epsilon_1\epsilon_0^lr \text{ for } l \in \{0,1,2\} \text{ and } k \in \mathbb{N}; \\ (13) \end{split}$$



Fig. 6: Transition probabilities of the different cells when the state f of the boundary belongs to S_1 . Precisely, the first line corresponds to $f \in \{(0, 1, 0, 0), (1, 1, 0, 0)\}$, the second line to $f \in \{0, 1\}^2 \times \{1\} \times \{0\}$, and the third line to $f \in \{0, 1\}^3 \times \{1\}$. The boundary position j depends on the case, but the modified boundary \tilde{j} is always the same.



Fig. 7: Transition probabilities when the boundary is in state (1, 0, 0, 0), resp. (0, 0, 0, 0) and \tilde{j} the position of the modified boundary.

We obtain then the following formula for the mean increment in two time steps when $f_t = (1, 0, 0, 0)$:

$$\mathbb{E}\left[\tilde{J}_{t} + \tilde{J}_{t+1}|f_{t} = (1,0,0,0)\right]$$

= $\mathbb{E}\left[\tilde{J}_{t}|f_{t} = (1,0,0,0)\right] + \mathbb{P}\left(f_{t+1} \in S_{1}|f_{t} = (1,0,0,0)\right) \mathbb{E}\left[\tilde{J}_{t+1}|f_{t+1} \in S_{1}\right]$
+ $\mathbb{P}\left(f_{t+1} = (1,0,0,0)|f_{t} = (1,0,0,0)\right) \mathbb{E}\left[\tilde{J}_{t+1}|f_{t+1} = (1,0,0,0)\right]$
+ $\mathbb{P}\left(f_{t+1} = (0,0,0,0)|f_{t} = (1,0,0,0)\right) \mathbb{E}\left[\tilde{J}_{t+1}|f_{t+1} = (0,0,0,0)\right]$ (14)

From Equations (12) and (13), we have computed or we can compute all the previous terms except $\mathbb{E}\left[\tilde{J}_{t+1}|f_{t+1}=(0,0,0,0)\right]$. In particular, with the help of a computer algebra system, one finds

$$\mathbb{E}\left[\tilde{J}_t|f_t = (1,0,0,0)\right] = -1 - \epsilon_0 + \epsilon_1 + \epsilon_0^2 + \epsilon_1^2 + 2\epsilon_0\epsilon_1 + 2\epsilon_0^3 + 2\epsilon_0\epsilon_1^2 - \epsilon_0^4 - \epsilon_0\epsilon_1^3 + \frac{(1-r)^3}{r}$$
(15)

For the missing term, we have to look at the transitions when $f_t = (0, 0, 0, 0)$.

Case $f_t = (0, 0, 0, 0)$: If $\tilde{j}_t = j$ and $f_t = (0, 0, 0, 0)$, then

$$\begin{split} (\tilde{j}_{t+1}, f_{t+1}) = \\ & \begin{cases} (\geq \tilde{j} - 3, (1, *, 0, 0)) & \text{w.p. } \epsilon_1 r \epsilon_0^2 r; \\ (\tilde{j} - 2 + k, (*, 0, 0, 0)) & \text{w.p. } r \epsilon_0^3 r \text{ for } k \in \{0, 1\}; \\ (\tilde{j} - 2 + k, (*, 0, 0, 0)) & \text{w.p. } r \epsilon_0^3 r \text{ for } k \in \{0, 1\}; \\ (\tilde{j} - 3, \{0, 1, *\} \times \{1\} \times \{0\}^2) & \text{w.p. } (1 - \epsilon_1) r \epsilon_0^2 r; \\ (\tilde{j} - 2, \{0, 1, *\} \times \{1\} \times \{0\}^2) & \text{w.p. } (1 - \epsilon_0) \epsilon_0^2 r; \\ (\tilde{j} - 2 + k, (0, 0, 0, 0)) & \text{w.p. } \epsilon_0^4 r \text{ for } k \in \{0, 1, 2, 3\}; \\ (\tilde{j} - 2 + k, (1, 0, 0, 0)) & \text{w.p. } \epsilon_1 \epsilon_0^3 r \text{ for } k \in \{0, 1\}; \\ (\tilde{j} - 2 + k, (1, 0, 0, 0)) & \text{w.p. } (1 - \epsilon_0) \epsilon_0^3 r \text{ for } k \in \{2, 3\}; \\ (\tilde{j} - 1, \{0, 1, *\}^2 \times \{0, 1\} \times \{1\}) & \text{w.p. } (1 - \epsilon_0) \epsilon_0 r; \\ (\tilde{j} - 1, \{0, 1, *\}^2 \times \{1\} \times \{0\}) & \text{w.p. } (1 - \epsilon_0) \epsilon_0 r; \\ (\tilde{j} - 1, \{0, 1, *\} \times \{0, 1\} \times \{1\} \times \{0\}) & \text{w.p. } (1 - \epsilon_0) \epsilon_0 r; \\ (\tilde{j} + 1 + k, \{0, 1\}^3 \times \{1\}) & \text{w.p. } (\epsilon_0 + \epsilon_1)^{k+1} \epsilon_1 r \text{ for } k \in \{0, 1\}; \\ (\tilde{j} - 1, \{0, 1\} \times \{1\} \times \{0\}) & \text{w.p. } (1 - \epsilon_0) \epsilon_0 r; \\ (\tilde{j} - 1, \{0, 1\} \times \{1\} \times \{0\}) & \text{w.p. } (\epsilon_0 + \epsilon_1)^{k+1} \epsilon_1 r \text{ for } k \in \{0, 1\}; \\ (\tilde{j} - 1, \{0, 1\} \times \{1\} \times \{0\}) & \text{w.p. } (1 - \epsilon_0) \epsilon_0^2 r \\ (\tilde{j}, \{0, 1\} \times \{1\} \times \{0\}^2) & \text{w.p. } (1 - \epsilon_0) \epsilon_0^2 r \\ (\tilde{j}, \{0, 1\} \times \{1\} \times \{0\}^2) & \text{w.p. } (1 - \epsilon_0) \epsilon_0^2 r \\ (\tilde{j}, 4 - 1, k, 1\} \times \{1\} \times \{0\}^2) & \text{w.p. } (\epsilon_0 + \epsilon_1)^{k+1} \epsilon_1 r \text{ for } k \in \{0, 1\}; \\ (\tilde{j} - 1, \{0, 1\} \times \{1\} \times \{0\}^2) & \text{w.p. } (1 - \epsilon_0) \epsilon_0^2 r \\ (\tilde{j}, \{0, 1\} \times \{1\} \times \{0\}^2) & \text{w.p. } (\epsilon_0 + \epsilon_1)^{k+1} \epsilon_0^2 r \text{ for } k \in \{0, 1\}; \\ (\tilde{j} + 2 + k, (0, 0, 0)) & \text{w.p. } (\epsilon_0 + \epsilon_1)^k \epsilon_0^4 r \text{ for } k \in \mathbb{N}; \\ (\tilde{j} + 2 + k, (1, 0, 0, 0)) & \text{w.p. } (\epsilon_0 + \epsilon_1)^k \epsilon_0^2 r \text{ for } k \in \mathbb{N}; \\ (\tilde{j} + 3 - l + k, \{0, 1\}^{4-l} \times \{1\} \times \{0\}^l) & \text{w.p. } (\epsilon_0 + \epsilon_1)^{k+3-l} \epsilon_1 \epsilon_0^l r \text{ for } l \in \{0, 1, 2\} \text{ and } k \in \mathbb{N}. \\ \end{split}$$

This allows to find, with the help of a computer algebra system, the following lower bound, denoted by I_0 , for $\mathbb{E}\left[\tilde{J}_t|f_t = (0,0,0,0)\right]$:

$$\mathbb{E}\left[\tilde{J}_t|f_t = (0,0,0,0)\right] \ge -1 - \epsilon_0 + \epsilon_1 - \epsilon_0^2 + \epsilon_1^2 + 3\epsilon_0\epsilon_1 + 6\epsilon_0^3 + 2\epsilon_0^2\epsilon_1 - 3\epsilon_0^4 - 3\epsilon_0^3\epsilon_1 + \frac{(1-r)^3}{r} = I_0.$$
(17)

Unfortunetaly, the lower bound I_0 is not greater than -1 for any ϵ_0 and ϵ_1 .

Back to the lower bound of $\mathbb{E}\left[\tilde{J}_t + \tilde{J}_{t+1}|f_t = (1, 0, 0, 0)\right]$. By putting the lower bound I_0 in the Equation (14), see Appendix 6.2 for details, we find that

$$\mathbb{E}\left[\tilde{J}_t + \tilde{J}_{t+1}|f_t = (1,0,0,0)\right] > -2.$$
 (18)

To conclude the proof, we now prove that the same inequality holds for $f_t = (0, 0, 0, 0)$.

Lower bound of $\mathbb{E}\left[\tilde{J}_t + \tilde{J}_{t+1} | f_t = (0, 0, 0, 0)\right]$. Using Equation (16), we obtain

$$\mathbb{E}\left[\tilde{J}_{t} + \tilde{J}_{t+1}|f_{t} = (0, 0, 0, 0)\right]$$

= $\mathbb{E}\left[\tilde{J}_{t}|f_{t} = (0, 0, 0, 0)\right] + \epsilon_{1}\epsilon_{0}^{2}r^{2}\mathbb{E}\left[\tilde{J}_{t+1}|f_{t+1} = (1, *, 0, 0)\right]$
+ \cdots + $\left(\sum_{l=0}^{2}\sum_{k=0}^{\infty}(1-r)^{k+3-l}\epsilon_{0}^{l}\epsilon_{1}\epsilon_{0}^{l}r\right)\mathbb{E}\left[\tilde{J}_{t+1}|f_{t+1} \in S_{1}\right].$

Finally, there are three terms that are not known and that we have to bound by below. For shortness, we denote by E^* the set $\{0, *\} \times \{*\} \times \{0\}^2$. By considering all the possible types of boundary in each subset, similarly as in Equation (9), we obtain

$$\begin{split} & \mathbb{E}\left[\tilde{J}_{t}|f_{t}=(*,0,0,0)\right] \geq \min\left(\mathbb{E}\left[\tilde{J}_{t}|f_{t}=(0,0,0,0)\right], \mathbb{E}\left[\tilde{J}_{t}|f_{t}=(1,0,0,0)\right]\right); \\ & \mathbb{E}\left[\tilde{J}_{t}|f_{t}=(1,*,0,0)\right] \geq \min\left(\mathbb{E}\left[\tilde{J}_{t}|f_{t}=(1,0,0,0)\right], \mathbb{E}\left[\tilde{J}_{t}|f_{t}\in S_{1}\right]\right); \\ & \mathbb{E}\left[\tilde{J}_{t}|f_{t}\in E^{*}\right] \geq \min\left(\mathbb{E}\left[\tilde{J}_{t}|f_{t}=(0,0,0,0)\right], \mathbb{E}\left[\tilde{J}_{t}|f_{t}=(1,0,0,0)\right], \mathbb{E}\left[\tilde{J}_{t}|f_{t}\in S_{1}\right]\right) \end{split}$$

For any $(\epsilon_0, \epsilon_1) \in [0, \frac{1}{2}]^2$, we have, see Appendices 6.3 and 6.4, the two following inequalities:

$$\mathbb{E}\left[\tilde{J}_t | f_t \in S_1\right] \ge I_0 \text{ and } \mathbb{E}\left[\tilde{J}_t | f_t \in S_1\right] \ge \mathbb{E}\left[\tilde{J}_t | f_t = (1, 0, 0, 0)\right]$$

Hence,

$$\mathbb{E}\left[\tilde{J}_t|f_t = (*,0,0,0)\right] \ge \min\left(I_0, \mathbb{E}\left[\tilde{J}_t|f_t = (1,0,0,0)\right]\right);$$
$$\mathbb{E}\left[\tilde{J}_t|f_t = (1,*,0,0)\right] \ge \mathbb{E}\left[\tilde{J}_t|f_t = (1,0,0,0)\right];$$
$$\mathbb{E}\left[\tilde{J}_t|f_t \in E^*\right] \ge \min\left(I_0, \mathbb{E}\left[\tilde{J}_t|f_t = (1,0,0,0)\right]\right).$$

Finally, we get

$$\mathbb{E}\left[\tilde{J}_{t} + \tilde{J}_{t+1}|f_{t} = (0, 0, 0, 0)\right]$$

$$\geq I_{0} + \mathbb{P}\left(f_{t+1} = (0, 0, 0, 0)|f_{t} = (0, 0, 0, 0)\right)I_{0}$$

$$+ \mathbb{P}\left(f_{t+1} \in S_{1}|f_{t} = (0, 0, 0, 0)\right)\mathbb{E}\left[\tilde{J}_{t+1}|f_{t+1} \in S_{1}\right]$$

$$+ \mathbb{P}\left(f_{t+1} \in \{(1, 0, 0, 0), (1, *, 0, 0)\}|f_{t} = (0, 0, 0, 0)\right)\mathbb{E}\left[\tilde{J}_{t+1}|f_{t+1} = (1, 0, 0, 0)\right]$$

$$+ \mathbb{P}\left(f_{t+1} = (*, 0, 0, 0) \cup E^{*}|f_{t} = (0, 0, 0, 0)\right)\min\left(I_{0}, \mathbb{E}\left[\tilde{J}_{t+1}|f_{t+1} = (1, 0, 0, 0)\right]\right)$$
(19)

The value of min $(I_0, \mathbb{E}\left[\tilde{J}_{t+1}|f_{t+1} = (1,0,0,0)\right])$ can be the left or the right term, depending on the values of ϵ_0 and ϵ_1 . Nevertheless, in Appendix 6.5, we prove that in both cases it is greater than -2, and so

$$\mathbb{E}\left[\tilde{J}_t + \tilde{J}_{t+1}|f_t = (0, 0, 0, 0)\right] > -2.$$
 (20)

Combined with Equations (12) and (18), it implies Equation (2) when n = 3, and so Theorem 1 for n = 3.

5 Perspectives

We have proved the desired result for n = 2 and n = 3, by examining the behaviour of the boundaries of decorrelated islands. The ergodicity of the hard-core PCA for $n \ge 4$ is still an open question. Our method could be used to handle the case of larger values of n, but the computations are becoming increasingly complex with the size of the neighbourhood, so that the n = 4 case is already difficult to tackle. However, one can ask whether there could be an automated way of handling larger neighbourhoods, and perhaps also other classes of PCA.

Concerning specifically the hard-core PCA, it seems that for fixed ϵ_0 , the mean increment in 1 (and 2) time step(s) is an increasing function in ϵ_1 . If this is indeed the case, it could simplify the analysis.

References

 Bhasin, D., Karmakar, S., Podder, M., Roy, S.: On a class of PCA with size-3 neighborhood and their applications in percolation games. Electron. J. Probab. 28 (2023)

- 2. Bušić, A., Mairesse, J., Marcovici, I.: Probabilistic cellular automata, invariant measures, and perfect sampling. Adv. in Appl. Probab. **45**(4), 960–980 (2013)
- 3. Casse, J.: Ergodicity of some probabilistic cellular automata with binary alphabet via random walks. Electronic Journal of Probability **28**, 1–17 (2023)
- Gács, P.: Reliable Cellular Automata with Self-Organization. Journal of Statistical Physics 103(1), 45–267 (2001)
- Holroyd, A.E., Marcovici, I., Martin, J.B.: Percolation games, probabilistic cellular automata, and the hard-core model. Probab. Theory Related Fields 174(3-4), 1187– 1217 (2019)
- Mairesse, J., Marcovici, I.: Around probabilistic cellular automata. Theoretical Computer Science 559, 42–72 (2014)
- Marcovici, I., Sablik, M., Taati, S.: Ergodicity of some classes of cellular automata subject to noise. Electron. J. Probab. 24 (2019)
- Toom, A.L., Vasilyev, N.B., Stavskaya, O.N., Mityushin, L.G., Kuryumov, G.L., Pirogov, S.A.: Discrete local Markov systems. In: Dobrushin, R.L., Kryukov, V.I., Toom, A.L. (eds.) Stochastic cellular systems: ergodicity, memory, morphogenesis. Manchester University Press (1990)

6 Appendices

6.1 Lower bound of $\mathbb{E}\left[ilde{J}_t + ilde{J}_{t+1} | f_t = (0,0) ight]$ for n=2

By Equation (7), Inequalities (8) and (9), and Equality (5), we obtain the following lower bound:

$$\mathbb{E}\left[\tilde{J}_t + \tilde{J}_{t+1}|f_t = (0,0)\right] \ge (1 + \epsilon_0 r^2 + 2\epsilon_0^2 r + \epsilon_0^2) \left(-\frac{1}{2}r^2 - 2\epsilon_0 r + \epsilon_0^2 r + \frac{1}{2}\epsilon_1^2 + \frac{(1-r)^2}{r}\right) + (1 - \epsilon_0 r^2 - 2\epsilon_0^2 r - \epsilon_0^2) \left(-\frac{1}{2}r + \epsilon_1 r + \frac{1}{2}\epsilon_0(1 - \epsilon_1)r + \frac{1}{2}\epsilon_1^2 r + \frac{1}{2}(1 - r)^2 + \frac{1}{2}\epsilon_1^2 + \frac{(1-r)^2}{r}\right).$$

Using a computer algebra system, we find the following polynomial expression in ϵ_0 and ϵ_1 :

$$\begin{split} & \mathbb{E}\left[\tilde{J}_{t} + \tilde{J}_{t+1} | f_{t} = (0,0)\right] \geq -1 + 2\frac{(1-r)^{2}}{r} \\ & + \underbrace{\epsilon_{0}^{6} + \frac{3}{2}\epsilon_{0}^{5}\epsilon_{1}}_{\geq 0} + \underbrace{\frac{7}{2}\epsilon_{0}^{4}}_{(1)} + \underbrace{6\epsilon_{0}^{3}\epsilon_{1}^{2}}_{(2)} + \underbrace{\frac{5}{2}\epsilon_{0}^{2}\epsilon_{1}^{3}}_{\geq 0} + \underbrace{4\epsilon_{0}^{2}\epsilon_{1}}_{(3)} + \underbrace{\frac{3}{2}\epsilon_{0}^{2}}_{(4)} + \underbrace{\frac{1}{2}\epsilon_{0}\epsilon_{1}^{5}}_{\geq 0} + \underbrace{\epsilon_{0}\epsilon_{1}^{3} + \epsilon_{0}\epsilon_{1}^{2}}_{(5)} + \underbrace{\frac{5}{2}\epsilon_{1}}_{(6)} \\ & - \left(\underbrace{\frac{7}{2}\epsilon_{0}^{5}}_{(1)} + \underbrace{\epsilon_{0}^{4}\epsilon_{1}^{2}}_{(2)} + \underbrace{\frac{3}{2}\epsilon_{0}^{4}\epsilon_{1}}_{(6)} + \underbrace{2\epsilon_{0}^{3}\epsilon_{1}^{3}}_{(2)} + \underbrace{4\epsilon_{0}^{3}\epsilon_{1}}_{(3)} + \underbrace{\frac{3}{2}\epsilon_{0}^{3}}_{(4)} + \frac{13}{2}\epsilon_{0}^{2}\epsilon_{1}^{2} + \underbrace{\frac{3}{2}\epsilon_{0}\epsilon_{1}^{4}}_{(5)} + \underbrace{\frac{1}{2}\epsilon_{0}\epsilon_{1}}_{(5)} \right). \end{split}$$

Now, we regroup terms of the form $c\epsilon_0^{\alpha_0}\epsilon_1^{\alpha_1}$ with terms of the form $-c\epsilon_0^{\beta_0}\epsilon_1^{\beta_1}$, with $\alpha_0 \leq \alpha_1$ and $\beta_0 \leq \beta_1$, in such a way that their sums are

$$c\epsilon_0^{\alpha_0}\epsilon_1^{\alpha_1}(1-\epsilon_0^{\beta_0-\alpha_0}\epsilon_1^{\beta_1-\alpha_1}) \ge 0$$

for any $\epsilon_0, \epsilon_1 \in [0, 1]$. We then obtain

 $\mathbb{E}\left[\tilde{J}_{t} + \tilde{J}_{t+1} | f_{t} = (0,0)\right] \ge -1 + 2\frac{(1-r)^{2}}{r} - \frac{13}{2}\epsilon_{0}^{2}\epsilon_{1}^{2} + \underbrace{\left(6-1-2\right)\epsilon_{0}^{3}\epsilon_{1}^{2}}_{(2)} + \underbrace{\left(1-\frac{1}{2}\right)\epsilon_{0}\epsilon_{1}^{2}}_{(2)} + \underbrace{\left(\frac{5}{2}-\frac{3}{2}-\frac{1}{2}\right)\epsilon_{1}}_{(2)}.$

$$(2) \qquad \underbrace{(5)}_{(5)} \qquad \underbrace{(6)}_{(6)}$$

To conclude, remark that

$$2\frac{(1-r)^2}{r} = 2(1-r)^2 + 2\frac{(1-r)^3}{r}$$
$$\geq 2(\epsilon_0 + \epsilon_1)^2 = 2\epsilon_0^2 + 4\epsilon_0\epsilon_1 + 2\epsilon_1^2.$$

This is enough to compensate the last negative term $-\frac{13}{2}\epsilon_0^2\epsilon_1^2$.

6.2 Lower bound of $\mathbb{E}\left[ilde{J}_t + ilde{J}_{t+1} | f_t = (1,0,0,0) ight]$ for n=3

By Equations (14), (13), (12), (15) and (17), we obtain the following lower bound:

$$\begin{split} & \mathbb{E}\left[\tilde{J}_t + \tilde{J}_{t+1} | f_t = (1,0,0,0)\right] \\ & \geq \mathbb{E}\left[\tilde{J}_t | f_t = (1,0,0,0)\right] + \mathbb{P}\left(f_{t+1} \in S_1 | f_t = (1,0,0,0)\right) \mathbb{E}\left[\tilde{J}_{t+1} | f_{t+1} \in S_1\right] \\ & + \mathbb{P}\left(f_{t+1} = (1,0,0,0) | f_t = (1,0,0,0)\right) \mathbb{E}\left[\tilde{J}_{t+1} | f_{t+1} = (1,0,0,0)\right] \\ & + \mathbb{P}\left(f_{t+1} = (0,0,0,0) | f_t = (1,0,0,0)\right) I_0. \end{split}$$

Now, replacing the value in ϵ_0 and ϵ_1 according to (13), (12), (15) and (17), and with the help of a computer algebra system, we obtain

$$\mathbb{E}\left[\tilde{J}_{t} + \tilde{J}_{t+1}|f_{t} = (1,0,0,0)\right] \geq -2 + 2\frac{(1-r)^{3}}{r} \\ + 3\epsilon_{1} + \underbrace{4\epsilon_{1}^{2}}_{(1)} + \epsilon_{1}^{4} + 2\epsilon_{0}\epsilon_{1} + 3\epsilon_{0}\epsilon_{1}^{2} + 5\epsilon_{0}\epsilon_{1}^{3} + \underbrace{14\epsilon_{0}\epsilon_{1}^{5}}_{(2)} + \epsilon_{0}\epsilon_{1}^{7} + 7\epsilon_{0}^{2}\epsilon_{1} + \underbrace{13\epsilon_{0}^{2}\epsilon_{1}^{3}}_{(3)} \\ + 10\epsilon_{0}^{2}\epsilon_{1}^{5} + \epsilon_{0}^{2}\epsilon_{1}^{7} + 11\epsilon_{0}^{3}\epsilon_{1} + \underbrace{26\epsilon_{0}^{3}\epsilon_{1}^{3}}_{(4)} + 2\epsilon_{0}^{3}\epsilon_{1}^{5} + \underbrace{3\epsilon_{0}^{4}}_{(5)+(6)} + \underbrace{49\epsilon_{0}^{4}\epsilon_{1}^{2}}_{(7)} + 18\epsilon_{0}^{4}\epsilon_{1}^{4} \\ + 12\epsilon_{0}^{5}\epsilon_{1} + 11\epsilon_{0}^{5}\epsilon_{1}^{3} + 2\epsilon_{0}^{6}\epsilon_{1}^{2} + \underbrace{3\epsilon_{0}^{7}}_{(6)} + 3\epsilon_{0}^{8}\epsilon_{1} + 2\epsilon_{0}^{9} \\ - \underbrace{(3\epsilon_{1}^{3}}_{(1)} + 15\epsilon_{0}\epsilon_{1}^{4} + \underbrace{6\epsilon_{0}\epsilon_{1}^{6}}_{(2)} + 13\epsilon_{0}^{2}\epsilon_{1}^{2} + \underbrace{12\epsilon_{0}^{2}\epsilon_{1}^{4}}_{(3)} + \underbrace{5\epsilon_{0}^{2}\epsilon_{1}^{6}}_{(2)} + \underbrace{5\epsilon_{0}^{2}\epsilon_{1}^{6}}_{(2)} + 25\epsilon_{0}^{3}\epsilon_{1}^{2} + \underbrace{12\epsilon_{0}^{3}\epsilon_{1}^{4}}_{(4)} + 24\epsilon_{0}^{4}\epsilon_{1} \\ + \underbrace{43\epsilon_{0}^{4}\epsilon_{1}^{3}}_{(7)} + \underbrace{3\epsilon_{0}^{4}\epsilon_{1}^{5}}_{(4)} + \underbrace{\epsilon_{0}^{5}}_{(5)} + 19\epsilon_{0}^{5}\epsilon_{1}^{2} + \underbrace{2\epsilon_{0}^{5}\epsilon_{1}^{4}}_{(5)} + \epsilon_{0}^{6}\epsilon_{1} + \epsilon_{0}^{6}\epsilon_{1}^{3} + 4\epsilon_{0}^{7}\epsilon_{1} + \underbrace{5\epsilon_{0}^{8}}_{(6)} \right]$$

As in Appendix 6.1, we regroup terms of the form $c\epsilon_0^{\alpha_0}\epsilon_1^{\alpha_1}$ with terms of the form $-c\epsilon_0^{\beta_0}\epsilon_1^{\beta_1}$, with $\alpha_0 \leq \alpha_1$ and $\beta_0 \leq \beta_1$. The last negative terms are then compensated by terms in

$$\frac{(1-r)^3}{r} = (\epsilon_0 + \epsilon_1)^3 + \dots + (\epsilon_0 + \epsilon_1)^8 + \frac{(1-r)^9}{r}.$$

Thus, $\mathbb{E}\left[\tilde{J}_t + \tilde{J}_{t+1} | f_t = (1, 0, 0, 0)\right] > -2.$

 $\textbf{6.3} \quad \text{Proof that} \ \mathbb{E}\left[\tilde{J}_t | f_t \in S_1\right] \geq I_0 \ \text{for} \ n=3$

Equations (12) and (17) give that

$$\mathbb{E}\left[\tilde{J}_{t}|f_{t} \in S_{1}\right] - I_{0}$$

$$= \epsilon_{0} + 2\epsilon_{1} + \epsilon_{0}^{2} + 3\epsilon_{1}^{2} - 3\epsilon_{1}^{3} + \epsilon_{0}\epsilon_{1} - \epsilon_{0}^{2}\epsilon_{1} + \epsilon_{1}^{4} + \epsilon_{0}^{3}\epsilon_{1}$$

$$- \left(-\epsilon_{0} + \epsilon_{1} - \epsilon_{0}^{2} + \epsilon_{1}^{2} + 3\epsilon_{0}\epsilon_{1} + 6\epsilon_{0}^{3} + 2\epsilon_{0}^{2}\epsilon_{1} - 3\epsilon_{0}^{4} - 3\epsilon_{0}^{3}\epsilon_{1}\right)$$

$$= 2\epsilon_{0} + \epsilon_{1} + 2\epsilon_{0}^{2} + 2\epsilon_{1}^{2} + \epsilon_{1}^{4} + 4\epsilon_{0}^{3}\epsilon_{1} + 3\epsilon_{0}^{4} - (3\epsilon_{1}^{3} + 2\epsilon_{0}\epsilon_{1} + 3\epsilon_{0}^{2}\epsilon_{1} + 6\epsilon_{0}^{3})$$

We use a remarkable identity to bound by below the underbraced terms. For the others negative terms, we use the fact that $0 \le \epsilon_0, \epsilon_1 \le 1/2$. Hence,

$$3\epsilon_1^3 \le 3\left(\frac{1}{2}\right)^2 \epsilon_1, \ 3\epsilon_0^2 \epsilon_1 \le \frac{3}{4}\epsilon_0, \text{ and } 6\epsilon_0^3 = 4\epsilon_0^3 + 2\epsilon_0^3 \le \epsilon_0 + \epsilon_0^2.$$

$$\mathbb{E}\left[\tilde{J}_{t}|f_{t} \in S_{1}\right] - I_{0}$$

$$\geq 2\epsilon_{0} + \epsilon_{1} + \underbrace{\epsilon_{0}^{2} + \epsilon_{1}^{2} + (\epsilon_{0} - \epsilon_{1})^{2}}_{4} + \epsilon_{1}^{4} + 4\epsilon_{0}^{3}\epsilon_{1} + 3\epsilon_{0}^{4} - \left(\frac{3}{4}\epsilon_{1} + \frac{3}{4}\epsilon_{0} + \epsilon_{0} + \epsilon_{0}^{2}\right)$$

$$= \frac{1}{4}\epsilon_{0} + \frac{1}{4}\epsilon_{1} + \epsilon_{1}^{2} + (\epsilon_{0} - \epsilon_{1})^{2} + \epsilon_{1}^{4} + 4\epsilon_{0}^{3}\epsilon_{1} + 3\epsilon_{0}^{4} > 0.$$

$$6.4 \quad \text{Proof that} \ \mathbb{E}\left[\tilde{J}_t | f_t \in S_1\right] \geq \mathbb{E}\left[\tilde{J}_t | f_t = (1,0,0,0)\right] \text{ for } n = 3$$

Equations (12) and (15) give that

$$\mathbb{E}\left[\tilde{J}_{t}|f_{t} \in S_{1}\right] - \mathbb{E}\left[\tilde{J}_{t}|f_{t} = (1,0,0,0)\right]$$

$$= \epsilon_{0} + 2\epsilon_{1} + \epsilon_{0}^{2} + 3\epsilon_{1}^{2} - 3\epsilon_{1}^{3} + \epsilon_{0}\epsilon_{1} - \epsilon_{0}^{2}\epsilon_{1} + \epsilon_{1}^{4} + \epsilon_{0}^{3}\epsilon_{1}$$

$$- (-\epsilon_{0} + \epsilon_{1} + \epsilon_{0}^{2} + \epsilon_{1}^{2} + 2\epsilon_{0}\epsilon_{1} + 2\epsilon_{0}^{3} + 2\epsilon_{0}\epsilon_{1}^{2} - \epsilon_{0}^{4} - \epsilon_{0}\epsilon_{1}^{3})$$

$$= 2\epsilon_{0} + \epsilon_{1} + 2\epsilon_{1}^{2} + \epsilon_{1}^{4} + \epsilon_{0}^{3}\epsilon_{1} + \epsilon_{0}^{4} + \epsilon_{0}\epsilon_{1}^{3}$$

$$- (3\epsilon_{1}^{3} + \epsilon_{0}\epsilon_{1} + \epsilon_{0}^{2}\epsilon_{1} + 2\epsilon_{0}^{3} + 2\epsilon_{0}\epsilon_{1}^{2}).$$

As in Appendix 6.3, the fact that $0 \le \epsilon_0, \epsilon_1 \le 1/2$ allows us to bound by below the negative terms.

$$\mathbb{E}\left[\tilde{J}_{t}|f_{t} \in S_{1}\right] - \mathbb{E}\left[\tilde{J}_{t}|f_{t} = (1,0,0,0)\right]$$

$$\geq 2\epsilon_{0} + \epsilon_{1} + 2\epsilon_{1}^{2} + \epsilon_{1}^{4} + \epsilon_{0}^{3}\epsilon_{1} + \epsilon_{0}^{4} + \epsilon_{0}\epsilon_{1}^{3} - \left(\frac{3}{4}\epsilon_{1} + \frac{1}{2}\epsilon_{0} + \frac{1}{4}\epsilon_{1} + \frac{1}{2}\epsilon_{0} + \frac{1}{2}\epsilon_{0}\right)$$

$$= \frac{1}{2}\epsilon_{0} + 2\epsilon_{1}^{2} + \epsilon_{1}^{4} + \epsilon_{0}^{3}\epsilon_{1} + \epsilon_{0}^{4} + \epsilon_{0}\epsilon_{1}^{3} > 0.$$

6.5 Lower bounds of
$$\mathbb{E}\left[ilde{J}_t + ilde{J}_{t+1} | f_t = (0,0,0,0)
ight]$$
 for $n=3$

When the minimum is I_0 : The lower bound of Equation (19) becomes

$$\begin{aligned} D_0 &= I_0 + \mathbb{P} \left(f_{t+1} = (0, 0, 0, 0) | f_t = (0, 0, 0, 0) \right) I_0 \\ &+ \mathbb{P} \left(f_{t+1} \in S_1 | f_t = (0, 0, 0, 0) \right) \mathbb{E} \left[\tilde{J}_{t+1} | f_{t+1} \in S_1 \right] \\ &+ \mathbb{P} \left(f_{t+1} \in \{ (1, 0, 0, 0), (1, *, 0, 0) \} | f_t = (0, 0, 0, 0) \right) \mathbb{E} \left[\tilde{J}_{t+1} | f_{t+1} = (1, 0, 0, 0) \right] \\ &+ \mathbb{P} \left(f_{t+1} = (*, 0, 0, 0) \cup E^* | f_t = (0, 0, 0, 0) \right) I_0 \end{aligned}$$

Now, replace the terms of this equation by their values given in Equations (17), (12), (15) and deduced from Equation (16). With the help of a computer algebra system, we obtain

$$\begin{split} D_0 &= -2 + 2 \frac{(1-r)^3}{r} \\ &+ 3\epsilon_1 + 4\epsilon_1^2 + \epsilon_1^4 + 4\epsilon_0\epsilon_1 + 6\epsilon_0^2\epsilon_1^3 + 5\epsilon_0^2\epsilon_1^5 + 4\epsilon_0^3 + 2\epsilon_0^3\epsilon_1 + 14\epsilon_0^3\epsilon_1^3 + 5\epsilon_0^3\epsilon_1^5 + 15\epsilon_0^4\epsilon_1 \\ &+ \epsilon_0^4\epsilon_1^4 + 10\epsilon_0^5 + 15\epsilon_0^5\epsilon_1^2 + 2\epsilon_0^5\epsilon_1^4 + 6\epsilon_0^6\epsilon_1 + 6\epsilon_0^7 + 3\epsilon_0^7\epsilon_1^2 + 8\epsilon_0^8\epsilon_1 + 4\epsilon_0^9 \\ &- (3\epsilon_1^3 + 9\epsilon_0^2\epsilon_1^4 + \epsilon_0^2\epsilon_1^6 + 10\epsilon_0^3\epsilon_1^2 + 11\epsilon_0^3\epsilon_1^4 + \epsilon_0^3\epsilon_1^6 + 9\epsilon_0^4 + 7\epsilon_0^4\epsilon_1^2 + \epsilon_0^4\epsilon_1^3 + 18\epsilon_0^5\epsilon_1 \\ &+ 8\epsilon_0^5\epsilon_1^3 + 3\epsilon_0^6 + 4\epsilon_0^6\epsilon_1^2 + 11\epsilon_0^7\epsilon_1 + 10\epsilon_0^8) \end{split}$$

We conclude that it is greater than -2 as in Appendices 6.1 and 6.2, by grouping terms and developing $(1-r)^3/r$ as much as necessary.

This technique does not hold for the term in ϵ_0^4 . Indeed, we obtain $6\epsilon_0^3 - 7\epsilon_0^4$ after that. To prove its positivity, we use the fact that $\epsilon_0 \leq 1/2$, and so $6\epsilon_0^3 - 7\epsilon_0^4 \geq (6 - 7/2)\epsilon_0^3 > 0$.

When the minimum is $\mathbb{E}\left[\tilde{J}_{t+1}|f_{t+1}=(1,0,0,0)\right]$: The lower bound of Equation (19) becomes

$$\begin{aligned} D_1 &= I_0 + \mathbb{P} \left(f_{t+1} = (0, 0, 0, 0) | f_t = (0, 0, 0, 0) \right) I_0 \\ &+ \mathbb{P} \left(f_{t+1} \in S_1 | f_t = (0, 0, 0, 0) \right) \mathbb{E} \left[\tilde{J}_{t+1} | f_{t+1} \in S_1 \right] \\ &+ \mathbb{P} \left(f_{t+1} \in \{ (1, 0, 0, 0), (1, *, 0, 0) \} | f_t = (0, 0, 0, 0) \right) \mathbb{E} \left[\tilde{J}_{t+1} | f_{t+1} = (1, 0, 0, 0) \right] \\ &+ \mathbb{P} \left(f_{t+1} = (*, 0, 0, 0) \cup E^* | f_t = (0, 0, 0, 0) \right) \mathbb{E} \left[\tilde{J}_{t+1} | f_{t+1} = (1, 0, 0, 0) \right] \end{aligned}$$

Now, replace the terms of this equation by their values given in Equations (17), (12), (15) and deduced from Equation (16). With the help of a com-

puter algebra system, we obtain

$$\begin{split} D_1 &= -2 + 2 \frac{(1-r)^3}{r} \\ &+ 3\epsilon_1 + 4\epsilon_1^2 + \epsilon_1^4 + 4\epsilon_0\epsilon_1 + 6\epsilon_0^2\epsilon_1^3 + 5\epsilon_0^2\epsilon_1^5 + 4\epsilon_0^3 + \epsilon_0^3\epsilon_1 + 4\epsilon_0^3\epsilon_1^3 + 7\epsilon_0^4\epsilon_1 + 5\epsilon_0^4\epsilon_1^2 \\ &+ 3\epsilon_0^4\epsilon_1^4 + 6\epsilon_0^5 + 14\epsilon_0^5\epsilon_1^3 + 10\epsilon_0^6\epsilon_1 + 22\epsilon_0^7 + 12\epsilon_0^7\epsilon_1^2 + 20\epsilon_0^8\epsilon_1 + 8\epsilon_0^9 \\ &- (3\epsilon_1^3 + 9\epsilon_0^2\epsilon_1^4 + \epsilon_0^2\epsilon_1^6 + 5\epsilon_0^3\epsilon_1^2 + \epsilon_0^3\epsilon_1^4 + 7\epsilon_0^4 + 9\epsilon_0^4\epsilon_1^3 \\ &+ 15\epsilon_0^5\epsilon_1^2 + 4\epsilon_0^5\epsilon_1^4 + 7\epsilon_0^6 + 4\epsilon_0^6\epsilon_1^3 + 36\epsilon_0^7\epsilon_1 + 24\epsilon_0^8). \end{split}$$

We conclude that it is greater than -2 as in Appendices 6.1 and 6.2, by grouping terms and developing $(1-r)^3/r$ as much as necessary.