

Mixing Times in Quasicrystals

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Journé-e nancé-i-en-ne de probabilité-e-s discrèt-e-s

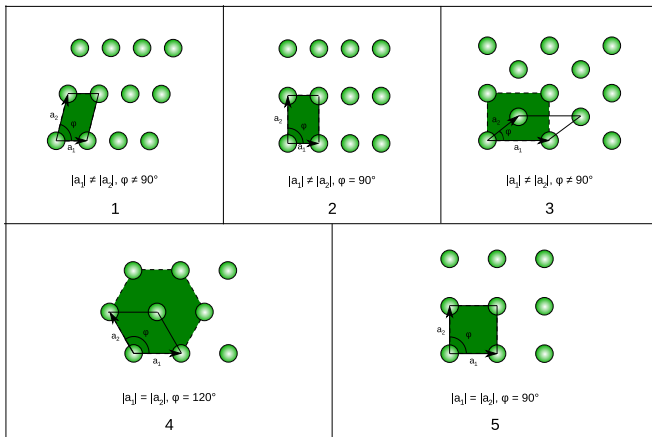
Outline

- 1 Quasicrystals
- 2 Modelization
- 3 Dimers
- 4 Beyond dimers

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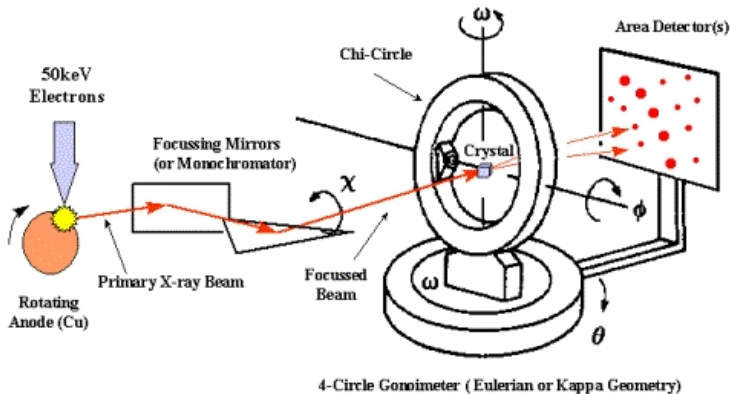
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Crystals (~ 1900)



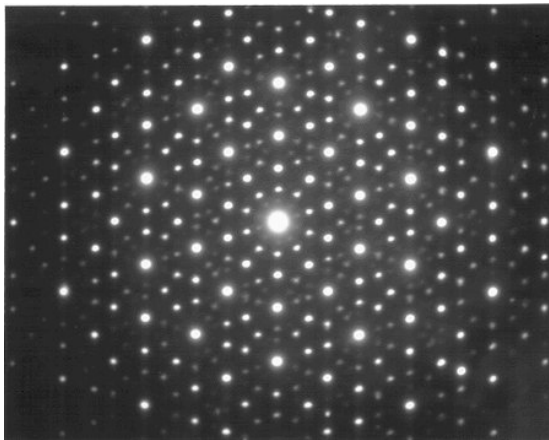
Crystal = ordered material = lattice + atomic pattern.

X-ray Diffraction (1912)



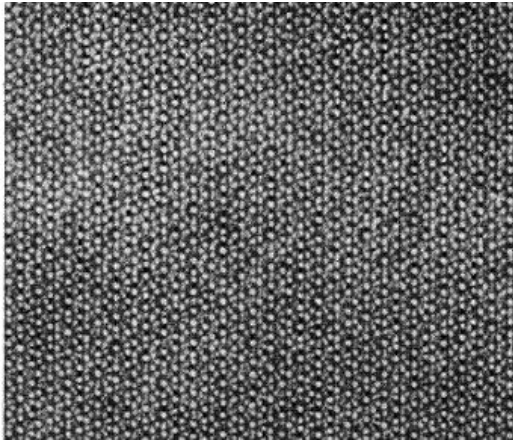
Crystal structure studied by X-ray diffraction.

Troubled times (1982–1992)



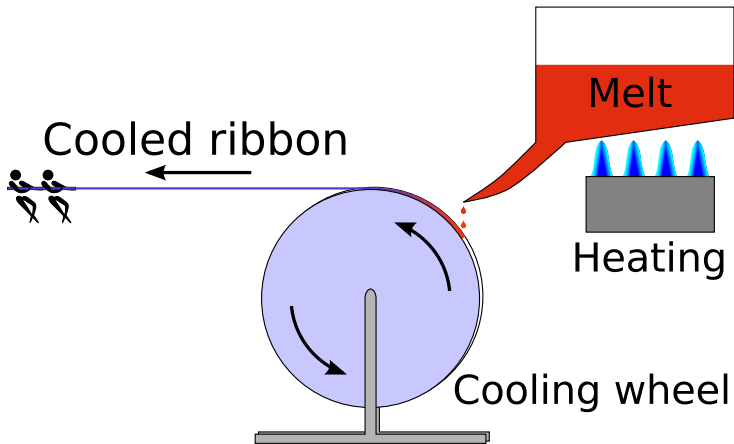
“Forbidden” ten-fold symmetry discovered \rightsquigarrow quasicrystals.

HRTEM



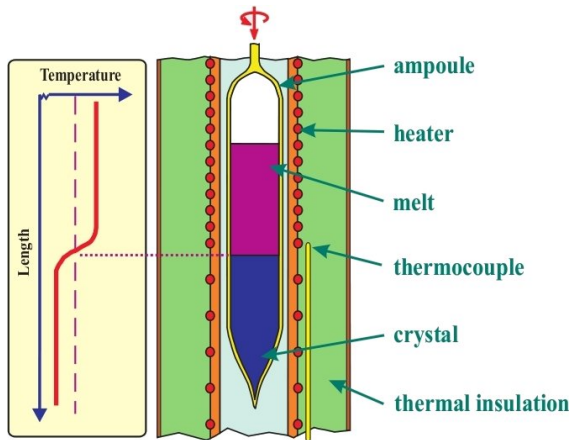
Transmission Electron Microscopy eventually showed the structure.

Quenching



First quasicrystals: rapid cooling from the melt. Many defects.

Bridgman-Stockbarger

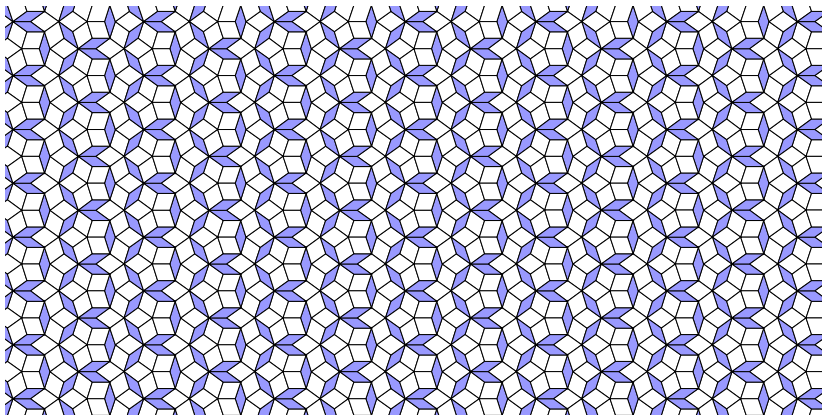


Today quasicrystals: slow cooling from the melt. Less defects.

Outline

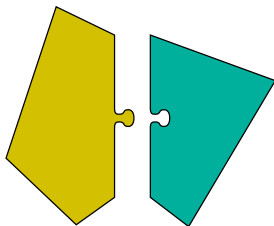
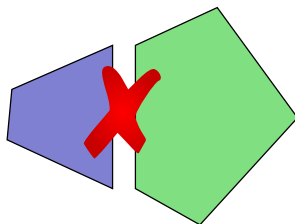
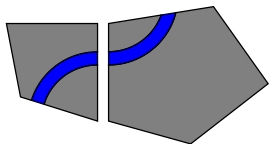
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Tilings



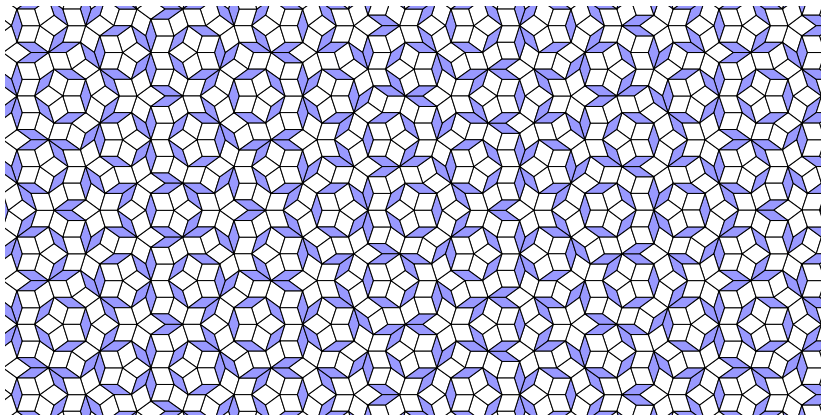
Covering of the space by interior-disjoint compacts called *tiles*.

Local rules



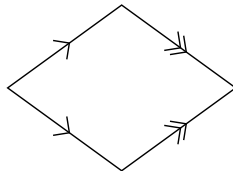
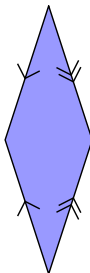
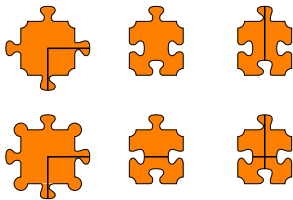
Inter-atomic energetic interaction \rightsquigarrow constraints on neighbor tiles.

Quasiperiodic tilings



A pattern reoccurs at uniformly bounded distance from any point.

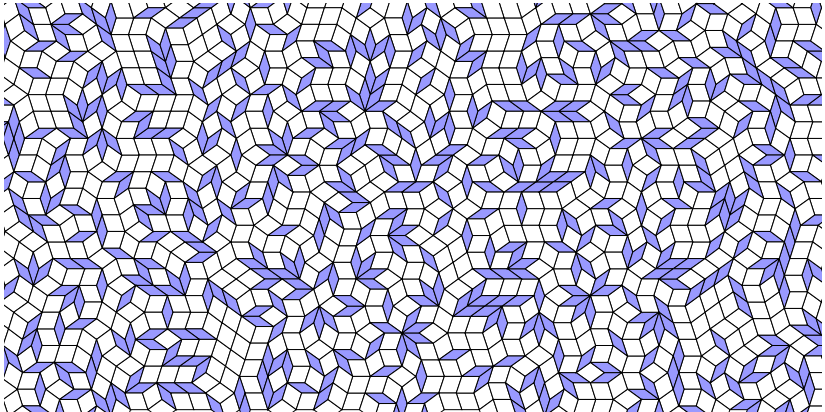
Aperiodic tilings (1964)



Aperiodic tile set: finite tile set that forms only non-periodic tilings.

It can always form quasiperiodic tilings.

Random tilings



At high T , entropy maximization supersedes energy minimization.

Cooling

General principle (*cf* Bridgman-Stockbarger method):

- start from an entropy maximizing tiling;
- perform local moves with proba. $\min(1, \exp(-\Delta E/T))$;
- progressively decrease T while still moving;
- hope to eventually minimize energy.

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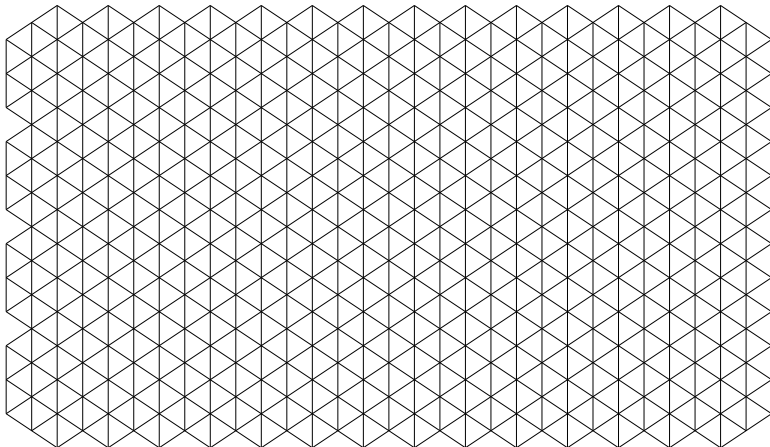
Directions:

- characterize entropy maximizing tilings;
- find suitable local moves;
- find a suitable cooling schedule;
- manage to prove something non-trivial. . .

Outline

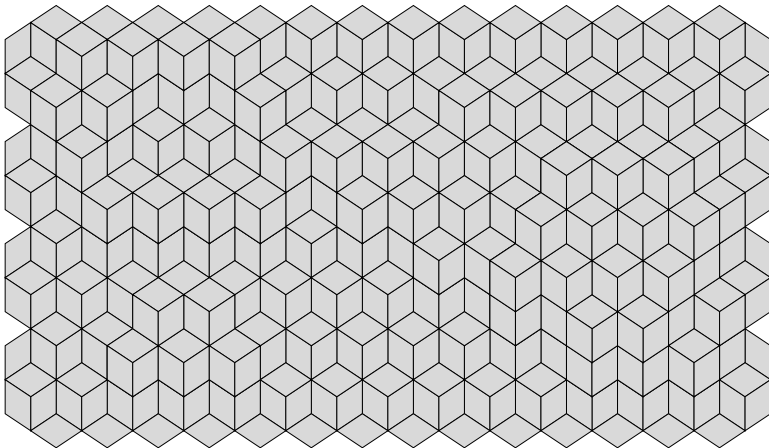
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Dimer tilings



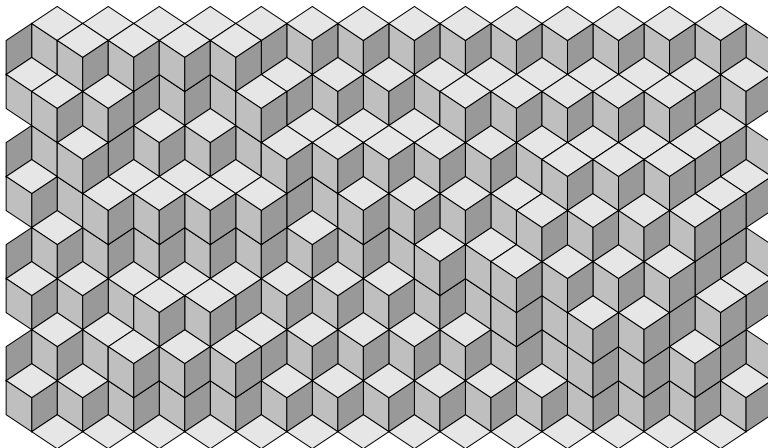
Perfect matchings of planar bipartite graphs.

Dimer tilings



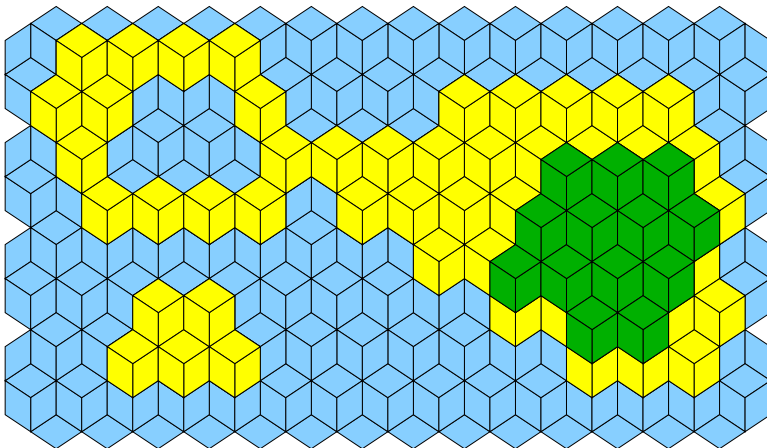
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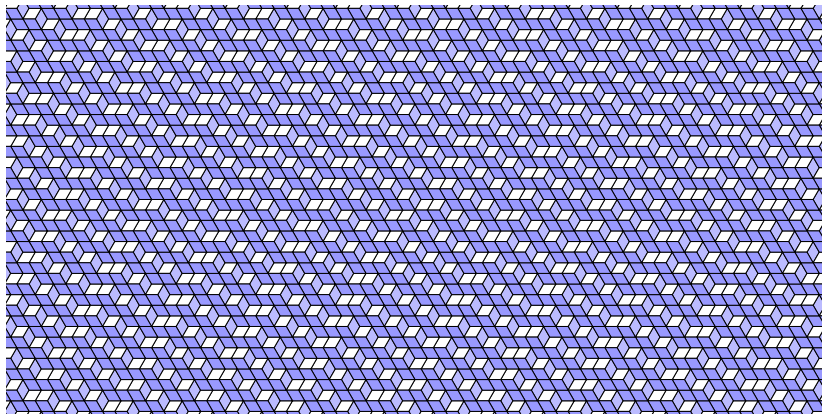
Perfect matchings of planar bipartite graphs. With height function.

Dimer tilings



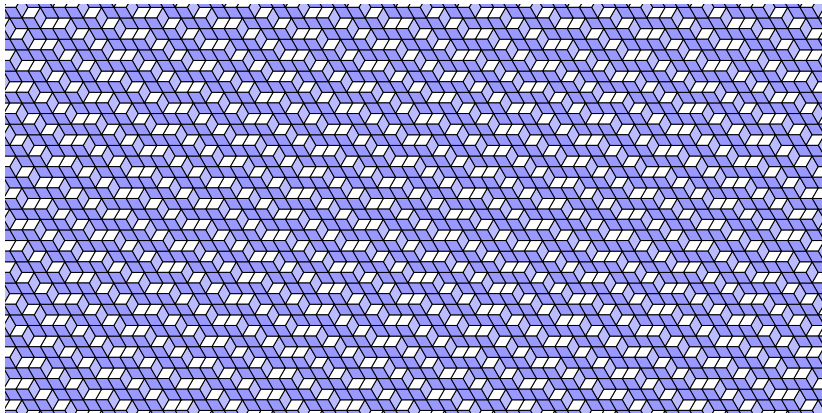
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Quasiperiodic tilings



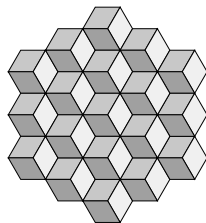
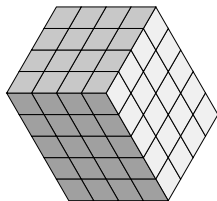
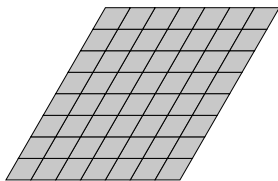
A tiling with lift in $P + [0, 1]^3$, P irrational plane, is quasiperiodic.

Aperiodic tilings



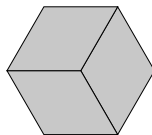
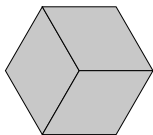
Tiles *must* be decorated (a thousand tiles seems not too much...)

Random tilings



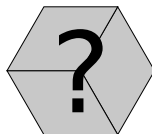
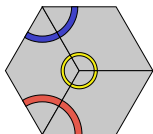
Entropy: $\frac{\log(\#\text{tilings})}{\#\text{tiles}}$. Maximum: Cohn-Kenyon-Propp, 2001.

Flip & local rules



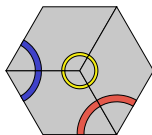
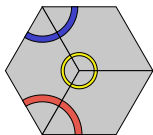
Classic local move on dimer tilings: *flip*.

Flip & local rules



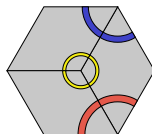
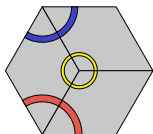
But how potential decorations are affected by performing a flip?

Flip & local rules



Is a flip a rotation?

Flip & local rules



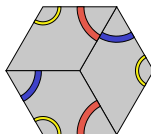
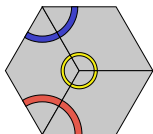
Is a flip a rotation? A reflexion?

Flip & local rules



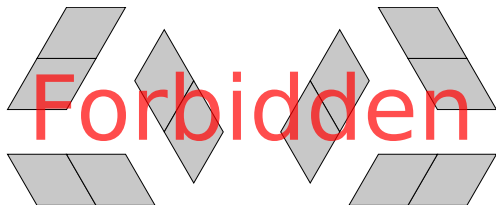
Is a flip a rotation? A reflexion? A tile exchange?

Flip & local rules



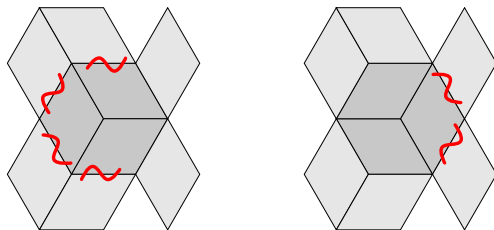
Is a flip a rotation? A reflexion? A tile exchange? A mix of this?

Flip & local rules



Problem avoided if the local rules are forbidden patterns.

Flip & local rules



To perform a flip, consider the forbidden patterns it is involved in.

Cooling

Perform flips which proba. $\min(1, \exp(-\Delta E/T))$. Mixing time τ ?

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Mostly studied case: infinite temperature (no local rules).

- $\tau = O(n^{3.5})$ tower-flips (Luby-Randall-Sinclair, 1995)
- $\tau = cn^2 \log(n)$ tower-flips (Wilson, 2004)
- $\tau = O(n^4 \log(n))$ flips (Randall-Tetali, 1999)

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and if the boundary is planar:

- $\tau = O(n^2 \log^c(n))$ flips (Caputo-Martinelli-Toninelli, 2011)
- $cn^2 \leq \tau \leq n^{2+o(1)}$ flips (Laslier-Toninelli, 2013)

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At the other extreme: zero temperature (non-increasing errors)

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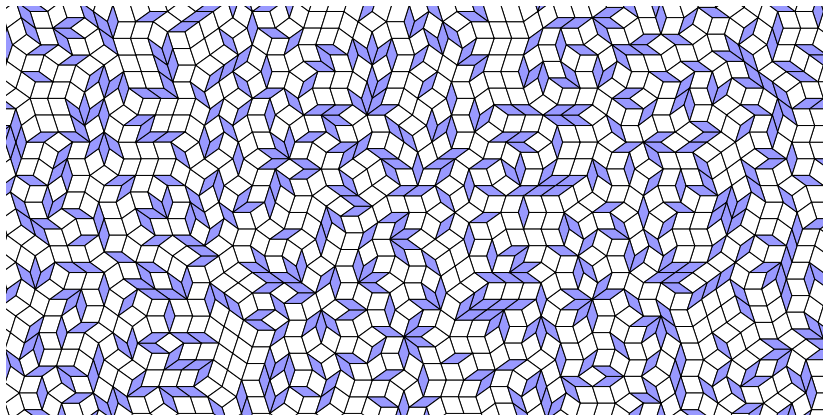
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Other temperatures? Cooling schedule?

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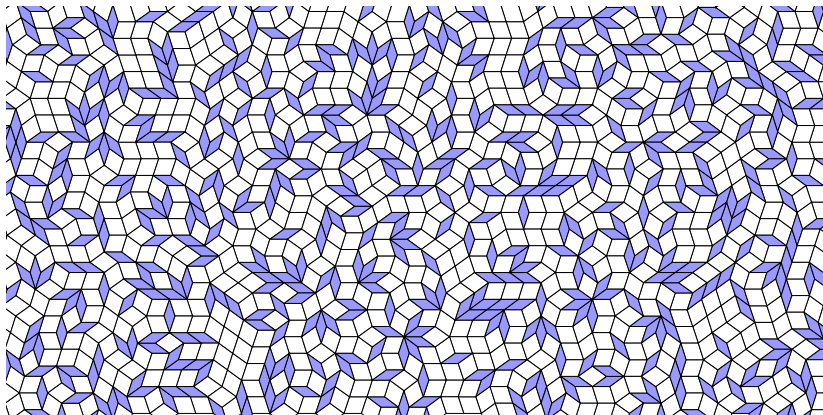
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Rhombus tilings



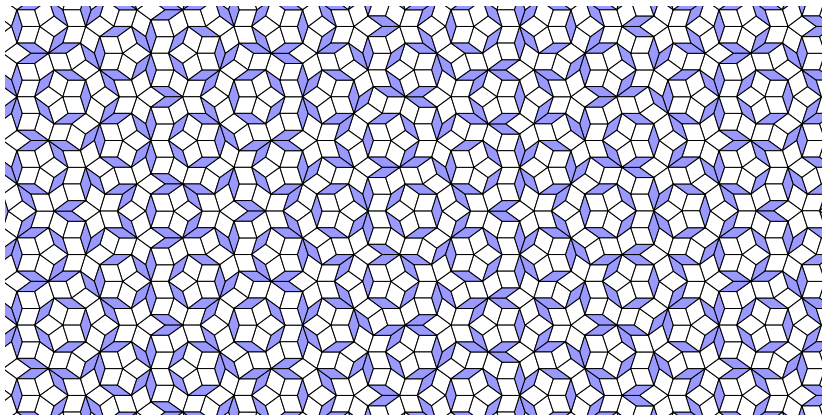
n non-colinear unit vectors in $\mathbb{R}^2 \rightsquigarrow \binom{n}{2}$ rhombi \rightsquigarrow tiling of \mathbb{R}^2 .

Rhombus tilings



Vectors as projections of a basis of $\mathbb{R}^n \rightsquigarrow$ tiling lifted in \mathbb{R}^n .

Rhombus tilings



A tiling with lift in $P + [0, 1]^n$, P irrational plane, is quasiperiodic.

Aperiodic tilings

When decorated tiles are allowed:

- local rules *iff* the plane is computable (F.-Sablik, 2012)

Aperiodic tilings

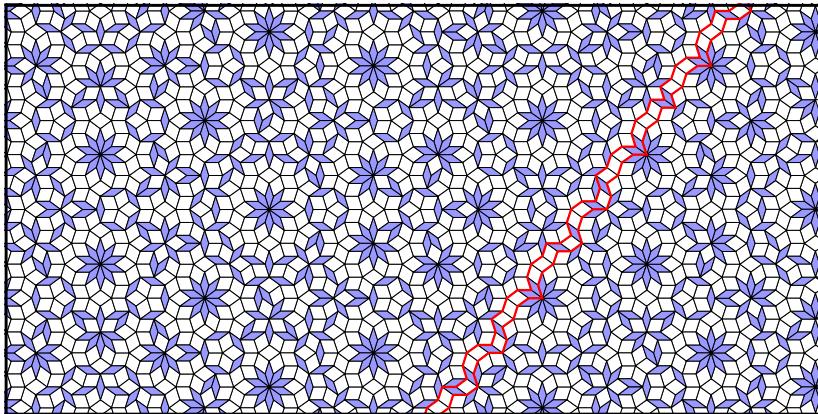
When decorated tiles are allowed:

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When only forbidden patterns are allowed:

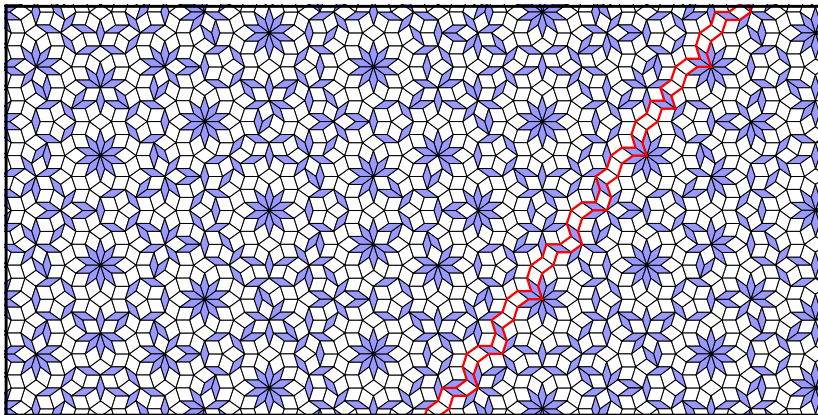
- no local rules for non-algebraic planes (Le, 1995)
- sufficient conditions (Levitov, Le, Socolar, Bédaride-F.)
- local rules *iff* the plane is characterized by its *subperiods*?

Example 1: (generalized) Penrose tilings (1974)



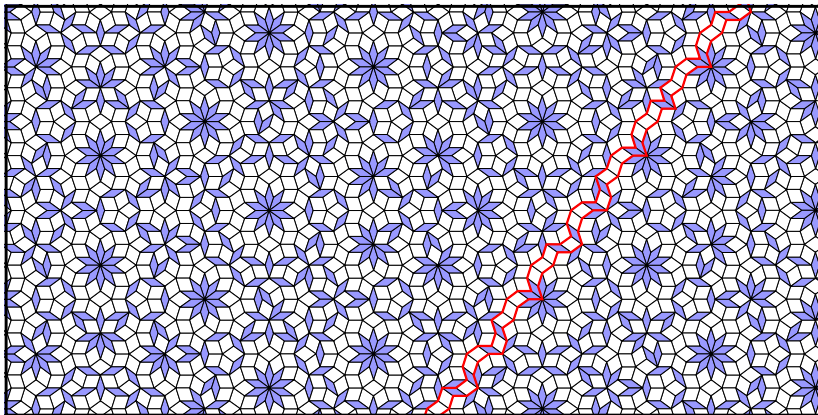
Lift in $P + [0, 1]^5$, $\vec{P} = \mathbb{R} \cos(2k\pi/5)_{0 \leq k < 5} + \mathbb{R} \sin(2k\pi/5)_{0 \leq k < 5}$.

Example 1: (generalized) Penrose tilings (1974)



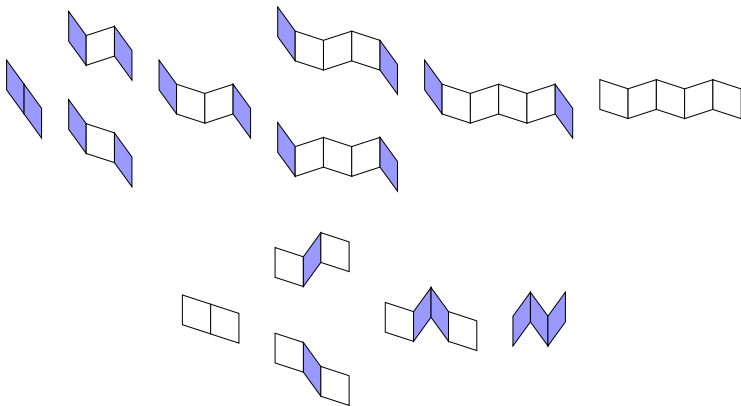
Lift in $P + [0, 1]^5$, $\vec{P} = (\varphi, 1, -1, -\varphi, \varphi, 1, -1, \varphi, 1, \varphi)$.

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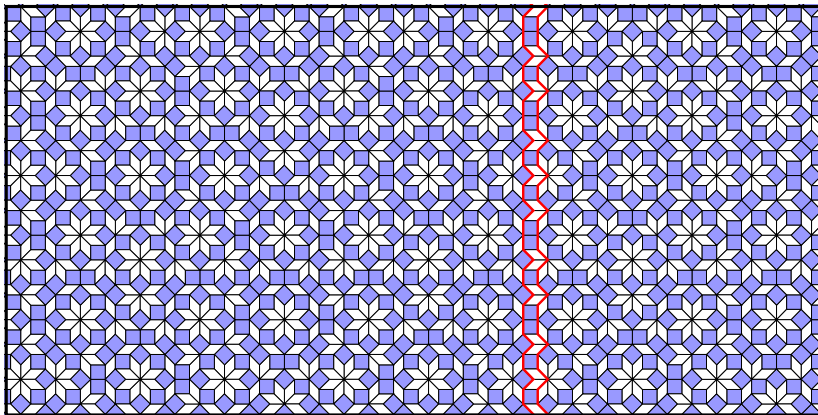
Characterized by an *alternation condition* (Socolar, 1990)

Example 1: (generalized) Penrose tilings (1974)



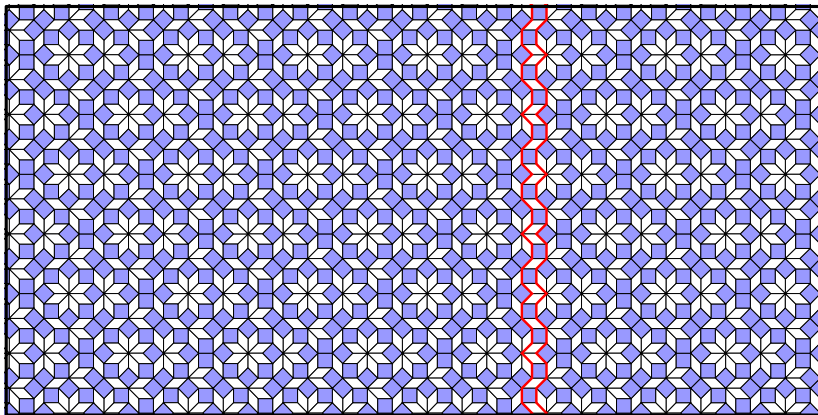
↪ admits local rules defined by finitely many forbidden patterns.

Example 2: Ammann-Beenker tilings (1970's-1982)



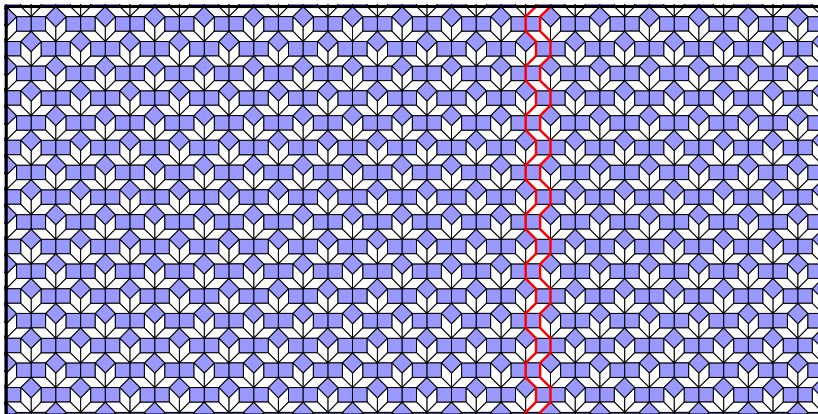
Lift in $P + [0, 1]^5$, $\vec{P} = (1, \sqrt{2}, 1, 1, \sqrt{2}, 1)$. Alternation condition?

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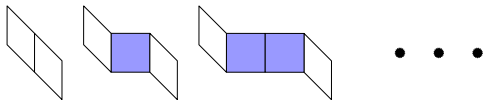
Enforces planarity, but allows any $\vec{P}_t = (1, t, 1, 1, 2/t, 1)$, $t > 0$.

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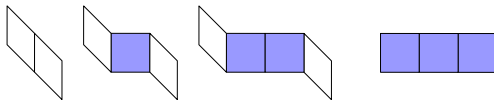
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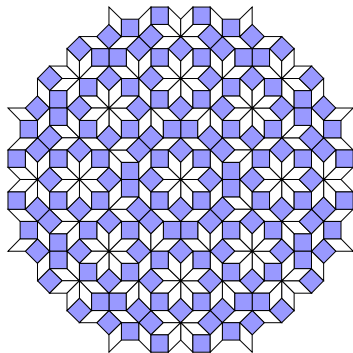
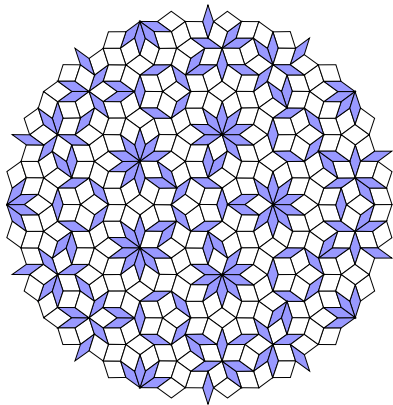
Forbidden patterns enforce planarity; boundary enforces the slope.

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Forbidden patterns enforce planarity; boundary enforces the slope.

Random tilings



Do Penrose or Ammann-Beenker tilings maximize entropy?

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Simulations suggest that τ is the same as for dimers:

- $\tau = O(n^2 \log(n))$ at $T = \infty$ for Beenker (Destainville, 2006)
- $\tau = O(n^2 \log(n))$ at $T = 0$ for Penrose & Beenker (F., 2009)

But there is no rigorous result. Even ergodicity is open for $T = 0$.

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Other temperatures? Cooling schedule?

Conclusion

Aperiodicity is much more interesting beyond dimer tilings...
...but random tilings or cooling process seem much complicated!