Cellular automata under noise: Empirical findings and applications

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Cellular automata

Cellular automata (CA) are completely discrete dynamical systems

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Interacting particle systems	D	D	C
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Components of a CA:

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Usually $x_i^{t+1} = [\Phi(\mathbf{x}^t)]_i = \phi_i(\mathbf{x}^t)$ depends only on a finite neighborhood of x_i^t and all $\phi_i(\,\cdot\,)$ are equal $\Longrightarrow$ rule table

Probabilistic cellular automata

If the rule Φ depends on a random variable, the CA becomes a probabilistic CA (PCA). We talk about the dynamics as “noisy”

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All PCA, no matter how they have been conceived, can be understood as being of this later type: a probabilistic mixture of CA

Example of a mixed PCA

ECA 184 with errors of "reading" type

Sometimes, with probability p, ECA 184 mistakes the central bit...
One obtains the following rule table for the resulting PCA:

	111	110	101	100	011	010	001	000
$1-p$	1	0	1	1	1	0	0	0
p	1	1	1	0	0	0	1	0

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First line: ECA 184 with probability $1-p$

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Second line: ECA 226 with probability p

We call the resulting rule PCA $(1-p)184$–$p226$

PCA dynamics and mean field approximation

The probability $P_t(\mathbf{x})$ of observing the PCA in state $\mathbf{x}$ at instant t given an initial distribution $P_0(\mathbf{x})$ is given by

$$P_{t+1}(\mathbf{x}') = \sum_{\mathbf{x}\in\Omega} \Phi(\mathbf{x}' \,|\, \mathbf{x}) P_t(\mathbf{x}),$$

$0 \leqslant \Phi(\mathbf{x}' \,|\, \mathbf{x}) \leqslant 1$ is the one-step conditional probability for $\mathbf{x} \to \mathbf{x}'$

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Since the cells are updated simultaneously and independently

$$\Phi(\mathbf{x}' \,|\, \mathbf{x}) = \prod_{i=1}^{L} \phi(x_i' \,|\, \mathbf{x})$$

PCA dynamics and mean field approximation

In $d = 1$ with $\Phi(\mathbf{x}' \,|\, \mathbf{x}) = \prod_i \phi(x_i' \,|\, x_{i-1}, x_i, x_{i+1})$ the dynamics of the marginal probability distribution $P_{t+1}(x)$ of observing a cell in state x at instant t obeys

$$P_{t+1}(x_i') = \sum_{x_{i-1}, x_i, x_{i+1}} \phi(x_i' \,|\, x_{i-1}, x_i, x_{i+1}) \, P_t(x_{i-1}, x_i, x_{i+1})$$

We see that $P_t(x_i)$ depends on $P_t(x_{i-1}, x_i, x_{i+1})$ which depend on $P_t(x_{i-2}, x_{i-1}, x_i, x_{i+1}, x_{i+2})$ and so on...

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Simplest approach to get a closed set of equations:

$$P_t(x_{i-1}, x_i, x_{i+1}) \approx P_t(x_{i-1}) P_t(x_i) P_t(x_{i+1})$$

a. k. a. single-cell mean field approximation

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The strugle for life

In the strugle for life—eating, moving, mating—two basic tenets are:

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  And the Lord God said, It is not good that the man should be alone; I will make him an help meet for him (Genesis 2:18)

The discrete logistic growth model

The dynamics of a single-species population subject to limiting resources can be described by the logistic map

$$x_{t+1} = x_t g(x_t) = r x_t \left(1 - \frac{x_t}{K}\right)$$

- $x_t \geqslant 0$ represents the size of the population

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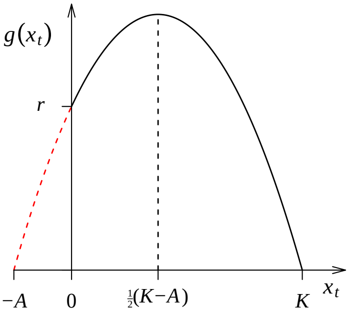
- $K > 0$ is the carrying capacity (maximum population viable under the given ecological conditions)
- $g(x_t) \geqslant 0$ is the intrinsic growth rate function of the population

Intrinsic growth rate function including weak Allee effects

Intrinsic growth rate function displaying combined features of logistic limitation to growth and weak Allee effect



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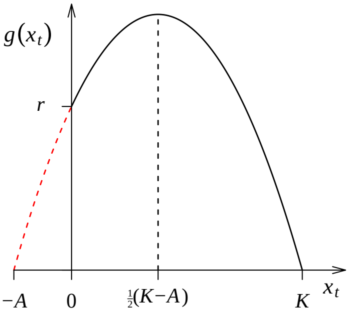

The simplest way to obtain this combined behavior is to extend the logistic map as

$$g^{(\pm)}(x_t) = r\left(1 - \frac{x_t}{K}\right)\left(\frac{x_t}{A} \pm 1\right),$$

where $0 < A < K$ represents a critical population threshold

Mixed PCA for single-species population dynamics

We want to propose a PCA to model the dynamics of a single-species population driven both by logistic growth and weak Allee effect:

- No spontaneous generation: $\phi(1 \,|\, 000) = 0$

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- Individuals that are neither lone nor in too packed a neighbourhood endure indefinitely: $\phi(1\,|\,110) = \phi(1\,|\,011) = 1$

Mixed PCA for single-species population dynamics

The mixed PCA embodying the rationale given before has rule table

	111	110	101	100	011	010	001	000
p	1	1	1	1	1	1	1	0
$1-p$	0	1	0	0	1	0	0	0

We obtained PCA $p254$–$(1-p)72$

Mean field solution

The single-cell mean field equation for PCA $p254$–$q72$ reads

$$x_{t+1} = px_t^3 + (2+p)x_t^2(1-x_t) + 3px_t(1-x_t)^2$$

with the expected structure for discrete-time models of population dynamics including Allee effects: $x_{t+1} = x_t g(x_t)$ with quadratic $g(x_t)$

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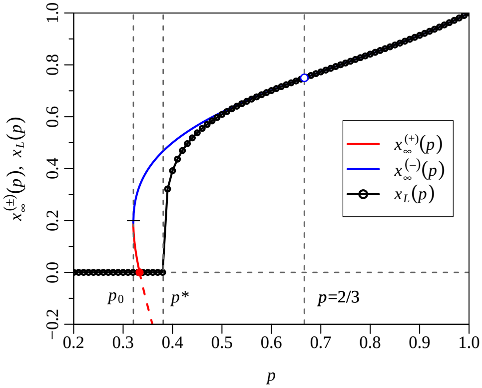
In the stationary state we must have $x_{t+1} = x_t \equiv x_\infty$ and the MF map has solutions $x_\infty^{(0)} = 0$ (absorbing state devoid of individuals) and

$$x_\infty^{(\pm)} = \frac{(5p-2) \pm \sqrt{(5p-2)^2 - 4(3p-2)(3p-1)}}{2(3p-2)}$$

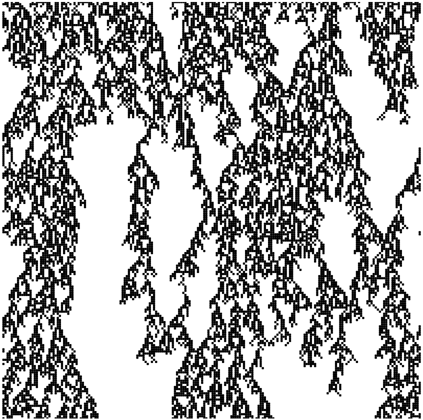
Mean field solution



Critical points and density profiles $\mathrm{Re}\{x_\infty^{(\pm)}(p)\}$ from the mean field solution and $x_L(p)$ from direct Monte Carlo simulations of PCA $p254$–$(1-p)72$ with $L = 10\,000$ cells averaged over 10 000 samples

Space-time diagram of PCA $p254$–$q72$

Space-time diagram of PCA $p254$–$q72$ at $p = 7/18 \simeq 0.389 > p^* \simeq 0.381$. $L = 200$ cells under p. b. c. evolved for 200 time steps (from top to bottom) from an initially random state of density $\sim 2/3$.



Other mixed PCA inspired by population dynamics

The most general left-right symmetric 1d "elementary PCA"

	111	110	101	100	011	010	001	000
0	$1-f$	$1-d$	$1-e$	$1-b$	$1-d$	$1-c$	$1-b$	$1-a$
1	f	d	e	b	d	c	b	a

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1	f	d	e	b	d	c	b	a

The single-cell mean field equation for this PCA reads ($x_t = P_t(1)$)

$$x_{t+1} = a(1-x_t)^3 + (2b+c)x_t(1-x_t)^2 + (2d+e)x_t^2(1-x_t) + fx_t^3$$

When do this equation corresponds to a population dynamics model?

Other mixed PCA inspired by population dynamics

We set $a = 0$, $u = 2b + c$ and $v = 2d + e$ and obtain $x_{t+1} = x_t g(x_t)$ with

$$g(x_t) = u + (v - 2u)x_t + (u - v + f)x_t^2$$

Case I – Quadratic logistic map
We must have $[1] > 0$, $[x_t] < 0$ and $[x_t^2] = 0$, leading to the conditions

$$u > 0, \quad -2u + v < 0, \quad u - v + f = 0$$

Case II – Cubic map with weak Allee effect
We must have $[1] > 0$, $[x_t] > 0$ and $[x_t^2] < 0$, leading to the conditions

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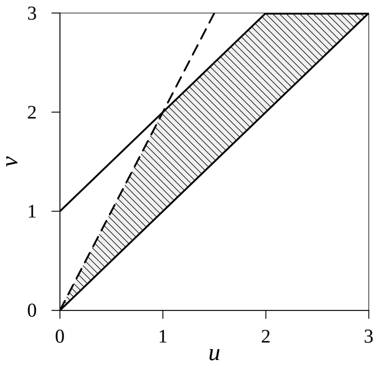
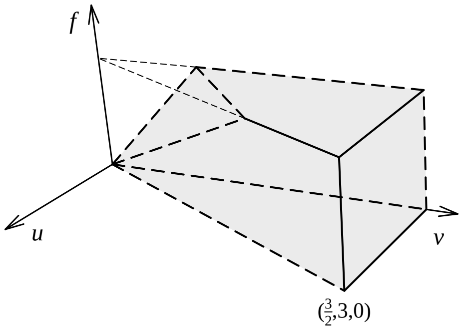
Solution sets



Projection on the *uv*-plane of the solution set for Case I



Solution set for Case II. The simplex has vertices at $(u,v,f) = (0,0,0)$, $(\frac{3}{2},3,0)$, $(0,3,0)$, $(1,2,1)$, $(\frac{3}{2},3,1)$, $(0,3,1)$, $(0,1,1)$

One-parameter solutions

If we parametrize $a, b, \ldots, f$ by a single parameter $p \in [0,1]$ we conclude that each become given either by p or by $1-p$

Imposing $a(p) = 0$ and taking symmetries into account reduce the number of possible cases to 16

One-parameter mixed PCA

Case I – Quadratic logistic map

In this case $f = (2d+e) - (2b+c) = f(b(p), c(p), d(p), e(p))$ and we have to examine only eight possible parametrizations for b, c, d, and e

The solution sets are given as line segments (equations in symmetic form) in the (u, v, f) space. In all cases $f = (2d+e) - (2b+c) = v - u$

(a,b,c,d,e,f)	$S_{\mathrm{I}}(a,b,c,d,e,f;p)$
$(0,p,p,p,p,*)$	$u/3 = v/3 \in (0,1], f = 0$
$(0,p,p,p,q,*)$	$u/3 = v-1 = (1-f)/2 \in (\frac{1}{5}, \frac{1}{2}]$
$(0,p,p,q,p,*)$	$u/3 = 2-v = (2-f)/4 \in (\frac{2}{7}, \frac{1}{2}]$
$(0,p,p,q,q,*)$	$u/3 = (3-v)/3 = (3-f)/6 \in (\frac{1}{3}, \frac{1}{2}]$
$(0,p,q,p,p,*)$	$u-1 = v/3 = (1+f)/2 \in [\frac{1}{2}, 1]$
$(0,p,q,p,q,*)$	$u-1 = v-1 \in [0,1], f = 0$
$(0,p,q,q,p,*)$	$u-1 = 2-v = (1-f)/2 \in (0, \frac{1}{2}]$
$(0,p,q,q,q,*)$	$u-1 = (3-v)/3 = (2-f)/4 \in [\frac{1}{4}, \frac{1}{2}]$

One-parameter mixed PCA

Case II – Cubic map with weak Allee effect

The solution sets are given as line segments (equations in symmetic form) in the (u, v, f) space

The last column gives the mixed rule specification of the PCA

(a, b, c, d, e, f)	$S_{\text{II}}(a, b, c, d, e, f; p)$	Mixed PCA
$(0, p, p, p, q, p)$	$u/3 = v - 1 = f \in (0, \frac{1}{5})$	$p222$–$q32$
$(0, p, p, q, p, p)$	$u/3 = 2 - v = f \in (0, \frac{2}{7})$	$p182$–$q72$
$(0, p, p, q, p, q)$	$u/3 = 2 - v = 1 - f \in (0, \frac{2}{7})$	$p54$–$q200$
$(0, p, p, q, q, p)$	$u/3 = (3 - v)/3 = f \in (0, \frac{1}{3})$	$p150$–$q104$
$(0, p, p, q, q, q)$	$u/3 = (3 - v)/3 = 1 - f \in (0, \frac{1}{3})$	$p22$–$q232$
$(0, p, q, q, q, p)$	$u - 1 = (3 - v)/3 = f \in [0, \frac{1}{5})$	$p146$–$q108$
$(0, p, q, q, q, q)$	$u - 1 = (3 - v)/3 = 1 - f \in [0, \frac{1}{5})$	$p18$–$q236$

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Scaling hypothesis and finite-size scaling

Mean field solutions, irrespective of the order employed, cannot capture the true nature of a phase transition $\Longrightarrow$ one needs exact solutions, RG calculations or numerical simulations

Scaling hypothesis

Close to a second-order phase transition the order parameter $x_L(t)$ of the PCA (its density of active cells) behaves like

$$x_L(t) \sim t^{-\beta/\nu_\parallel} \Phi(\varepsilon t^{1/\nu_\parallel}, t^{\nu_\perp/\nu_\parallel}/L)$$

where $\varepsilon = |p - p^*| \geqslant 0$ and L is the size of the array

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For $L \nearrow \infty$, $x_L(t) \sim t^{-\beta/\nu_\parallel}\Phi(\varepsilon t^{1/\nu_\parallel})$ with $\Phi(u \ll 1) \sim$ const and $\Phi(u \gg 1) \sim u^{\beta}$. Close to the critical point $\varepsilon \approx 0$ and for large L we must thus observe $x_L(t) \sim t^{-\delta}$, with $\delta = \beta/\nu_\parallel$

Directed percolation conjecture

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All phase transitions into an absorbing state in short-ranged systems in the absence of conserved quantities belong to the directed percolation universality class of critical behaviour

According to the DP conjecture, the critical behavior of all $1d$ single-component PCA belongs to the DP universality class

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Is it possible to find/devise mixed PCA in other universality classes?

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Ergodicity and density classification problem

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Ergodicity of $1d$ PCA

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Positive probabilities conjecture

$1d$ systems with short-range interactions and positive transition probabilities are always ergodic

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Positive probabilities conjecture

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Density classification problem
The task consists in classifying arrays of symbols according to their initial density using local rules, and is completed successfully if all the cells of the CA converge to the initial majority state in $O(L)$ time

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The four-state GKL model IV

GKL model IV is a four-state CA with $\omega = \{\rightarrow, \leftarrow, \uparrow, \downarrow\}$ and rules

$$\phi_{\mathrm{IV}}(\rightarrow, x_i, x_{i+1}) = \rightarrow, \qquad \text{if } x_i, x_{i+1} \neq \leftarrow$$

$$\phi_{\mathrm{IV}}(x_{i-1}, \rightarrow, x_{i+1}) = \begin{cases} \downarrow, & \text{if } x_{i-1} \in \{\leftarrow, \uparrow\} \\ \rightarrow, & \text{otherwise} \end{cases}$$

$$\phi_{\mathrm{IV}}(x_{i-1}, x_i, x_{i+1}) = \uparrow, \qquad \text{if } x_i \in \{\uparrow, \downarrow\} \text{ and first does not apply}$$

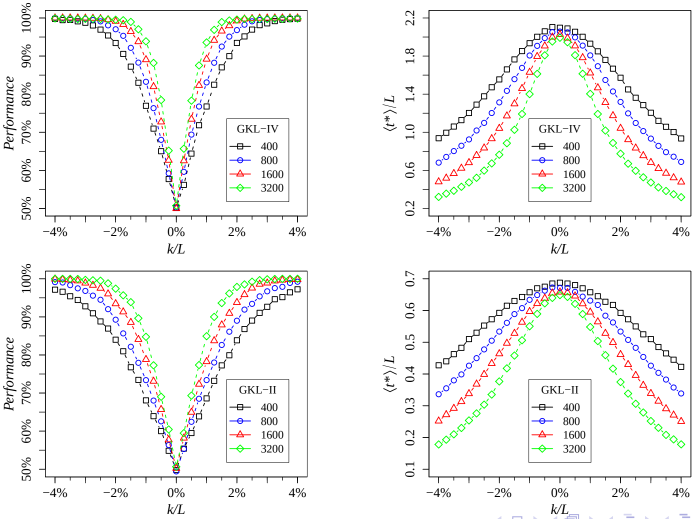
These rules define only 42 transitions. The missing 22 transitions are determined by the supplemental reflection rule

$$\phi_{\mathrm{IV}}(x_{i-1}, x_i, x_{i+1}) = \phi_{\mathrm{IV}}(x_{i+1}^*, x_i^*, x_{i-1}^*)^*,$$

with

$$\rightarrow^* = \leftarrow, \quad \leftarrow^* = \rightarrow, \quad \uparrow^* = \uparrow, \quad \downarrow^* = \downarrow$$

GKL-IV density classification performance



GKL-IV model under noise

GKL considered random writing errors: at every time step, with probability $1-\alpha$ the transition follows the deterministic rules and with probability α the final state is chosen uniformly at random

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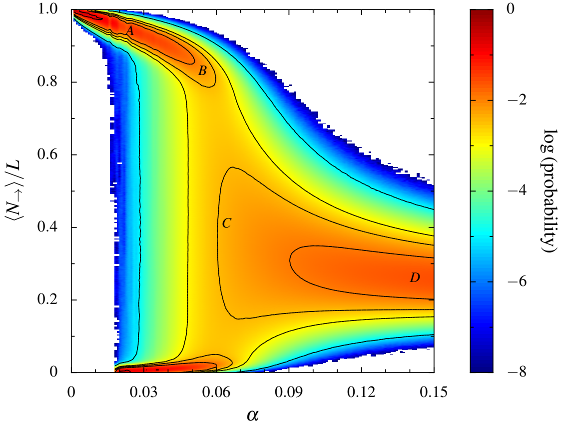
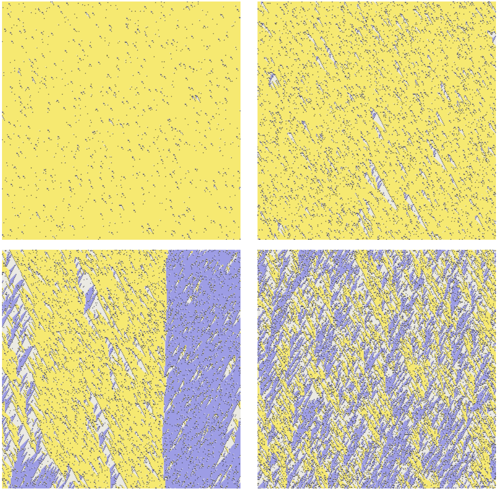
Is the noisy GKL-IV model ergodic?

Stationary pdf of the majority state in the GKL-IV PCA

Density plot and level curves of the pdf of the majority state in the stationary state of the GKL-IV PCA for an array of $L = 400$ cells



Space-time diagrams of the noisy GKL-IV



Ergodicity of the GKL-IV PCA

Based on an analogy between the flipping time $\tau(L,\alpha)$ between the majority phases of a PCA of L cells subject to noise level α and the correlation length $\xi_{\|}(L,T)$ of a 2D equilibrium interacting classical spin model of linear size L at temperature T we expect that

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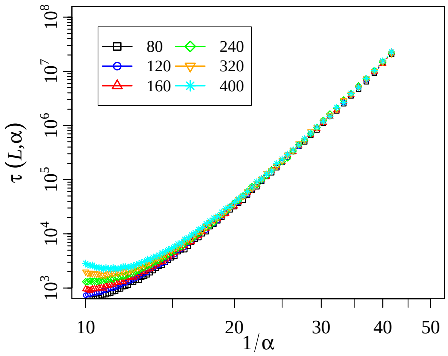
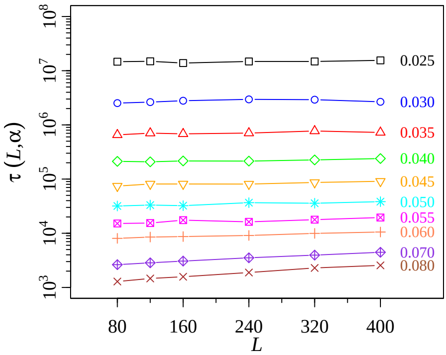
In a nonergodic phase $u(L,\alpha) \sim b(L)/\alpha$ for $\alpha \searrow 0$ and fixed L and $u(L,\alpha) \sim c(\alpha)L$ for fixed α and $L \nearrow \infty$

Ergodicity of the GKL-IV PCA





Flipping times $\tau(L,\alpha)$ grow exponentially as the PCA dynamics becomes less noisy, diverging as $\alpha \searrow 0$

While $\tau(L,\alpha)$ clearly diverges as $\alpha \searrow 0$, it does not so as L grows at least down to $\alpha = 0.025$

Summary on GKL-IV

- GKL-IV performs well in the density classification problem, with a performance comparable with that of GKL-II

Note that while the ergodicity of $1d$ deterministic CA is in general undecidable, most PCA are believed to be ergodic

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- On the negative side, GKL-IV takes longer (almost 3 times) to reach consensus
- We found signs of an ergodic-nonergodic phase transition at some small finite positive level of noise $\alpha \lesssim 0.016$
- GKL-IV PCA may be nonergodic but our data are inconclusive — conclusions are affected by the finite size of the system and the finite time of the simulations

Note that while the ergodicity of $1d$ deterministic CA is in general undecidable, most PCA are believed to be ergodic

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Population dynamics in $d \geqslant 2$

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- Good starting point: $2d$, single-species, one-parameter PCA $\Longrightarrow$ may already provide a model for invasion dynamics
- Alternatively, investigate $1d$, single-species, two-parameter PCA $\Longrightarrow$ possible interesting phase diagram (α, λ)

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Modified GKL models

We have found many GKL-like models capable of classifying density

- A modified GKL-II model involving neighborhoods of type i, $i \pm 1$, $i \pm (2k+1)$, $k > 1$ preserve the density classification capacity approx. at the same level but achieves consensus faster

$$\mathrm{GKL}(j,k):\; x_i^{t+1} = \begin{cases} \mathrm{maj}\left(x_{i-k}^t, x_{i-j}^t, x_i^t\right), & \text{if } x_i^t = 0 \\ \mathrm{maj}\left(x_i^t, x_{i+j}^t, x_{i+k}^t\right), & \text{if } x_i^t = 1 \end{cases}$$

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- We have found 4 and 5-state GKL-like eroders with “catalyst states” that deserve further investigation

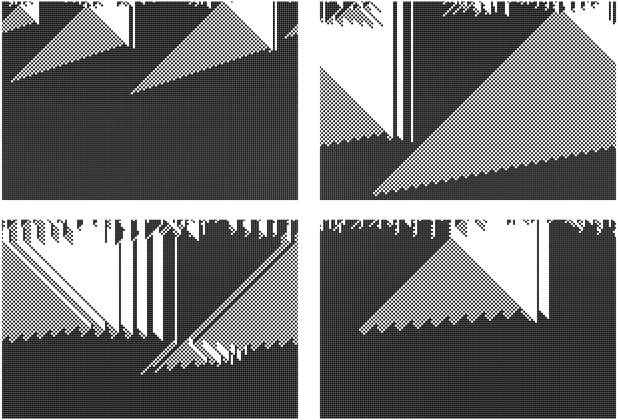
Modified GKL models

Density classification performance of GKL(j,k) for different (j,k) in an array of $L = 149$ cells averaged over 10^7 random initial conditions near the critical density $\rho^* = 1/2$

(j,k)	$(1,3)$	$(1,5)$	$(1,7)$	$(1,9)$	$(1,11)$
$\langle f \rangle$	81.5%	81.2%	81.5%	80.3%	76.0%
$\langle t^* \rangle / L$	0.576	0.383	0.289	0.280	0.413

Modified GKL models

Space-time patterns of GKL(j,k) with $L = 149$, $0 \leqslant t \leqslant 100$, and initial conditions near the critical density $\rho^* = 1/2$



In reading order $(j,k) = (1,3)$ (usual GKL-II), $(1,5)$, $(1,7)$, and $(1,9)$

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- An extinction-survival-type phase transition in the probabilistic cellular automaton $p182$–$q200$, *J. Phys. A: Math. Theor.* **44**, 155001 (2011) [with M. J. de Oliveira]

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Merci beaucoup!